Understanding of superfluid density in the inner crust of neutron stars

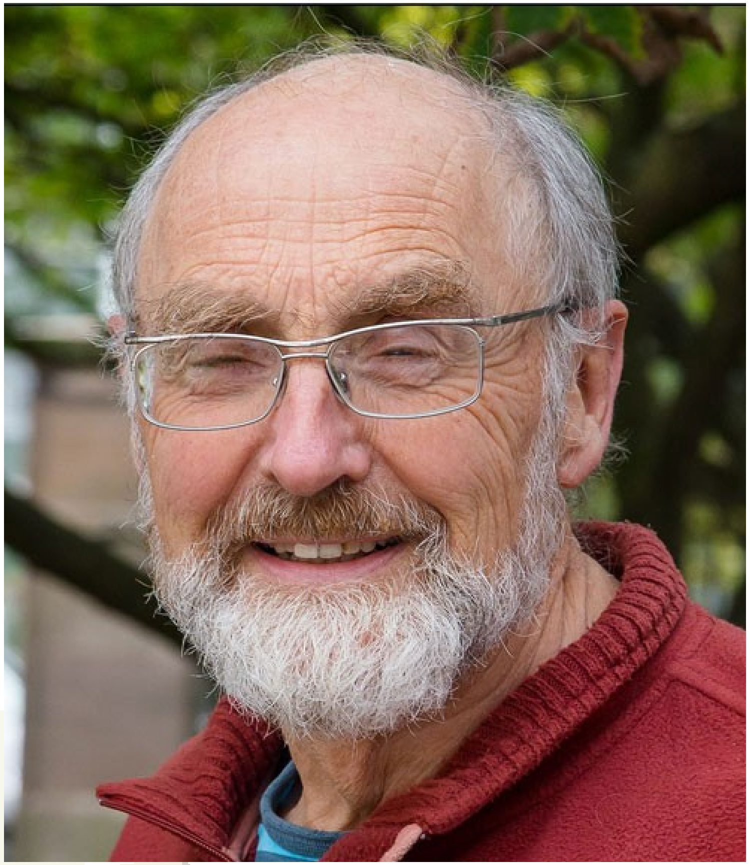
— competition between two "gaps"

GW & C. J. Pethick, PRL **119**, 062701 (2017).
Y. Minami & GW, PRR **4**, 033141 (2022).

Gentaro Watanabe
(渡边 元太郎)

(Zhejiang Univ.)

Collaborators



Christopher Pethick
(Niels Bohr Institute)

Yuki Minami (南佑樹)
(Kyoto Univ.)

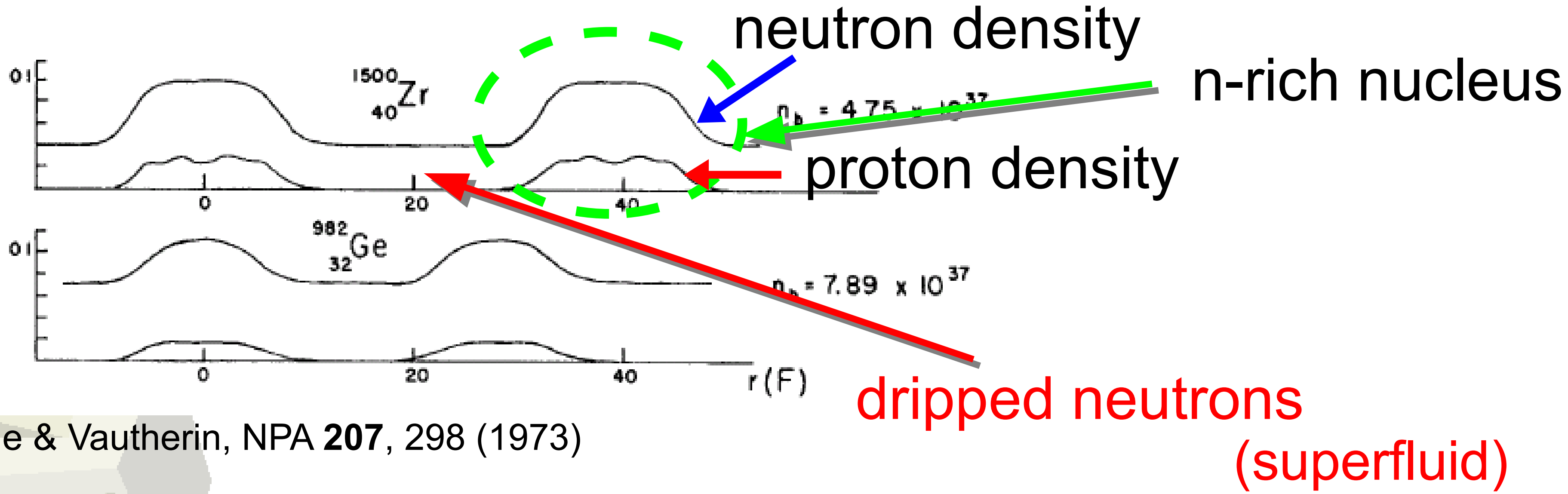
GW & C. J. Pethick, PRL **119**, 062701 (2017).

Y. Minami & GW, PRR **4**, 033141 (2022).

Neutron superfluid in NS crusts

In neutron-star (inner) crusts:

Dripped neutrons form a superfluid (by *s*-wave pairing).



Negele & Vautherin, NPA **207**, 298 (1973)

Origin of pairing int.:

Attractive component of nuclear force.

Low density neutron matter

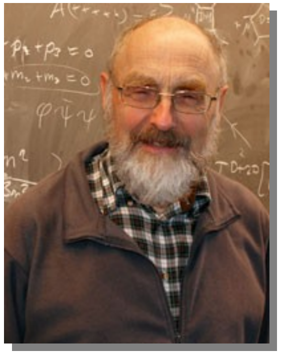
Dripped neutron gas in inner crusts:

Neutron matter at low densities

$(\rho << \rho_0)$

Great advantages over symmetric nuclear matter:

- Only 2 spin-isospin DOFs
- Non-self bound
- Pauli principle → 3-body force suppressed.



Pandharipande, Pethick, etc.

Similarity btwn. n-matter & cold atoms

Recent progress

Similarity btwn. low density n-matter in NS crusts & cold gases of fermionic atoms

- Theory

  - Improved many-body calc.

    QMC calc. for neutron matter & cold atomic gases

    e.g., Gandolfi, Gezerlis & Carlson, Annu. Rev. Nucl. Part. Sci. **65**, 303 (2015)

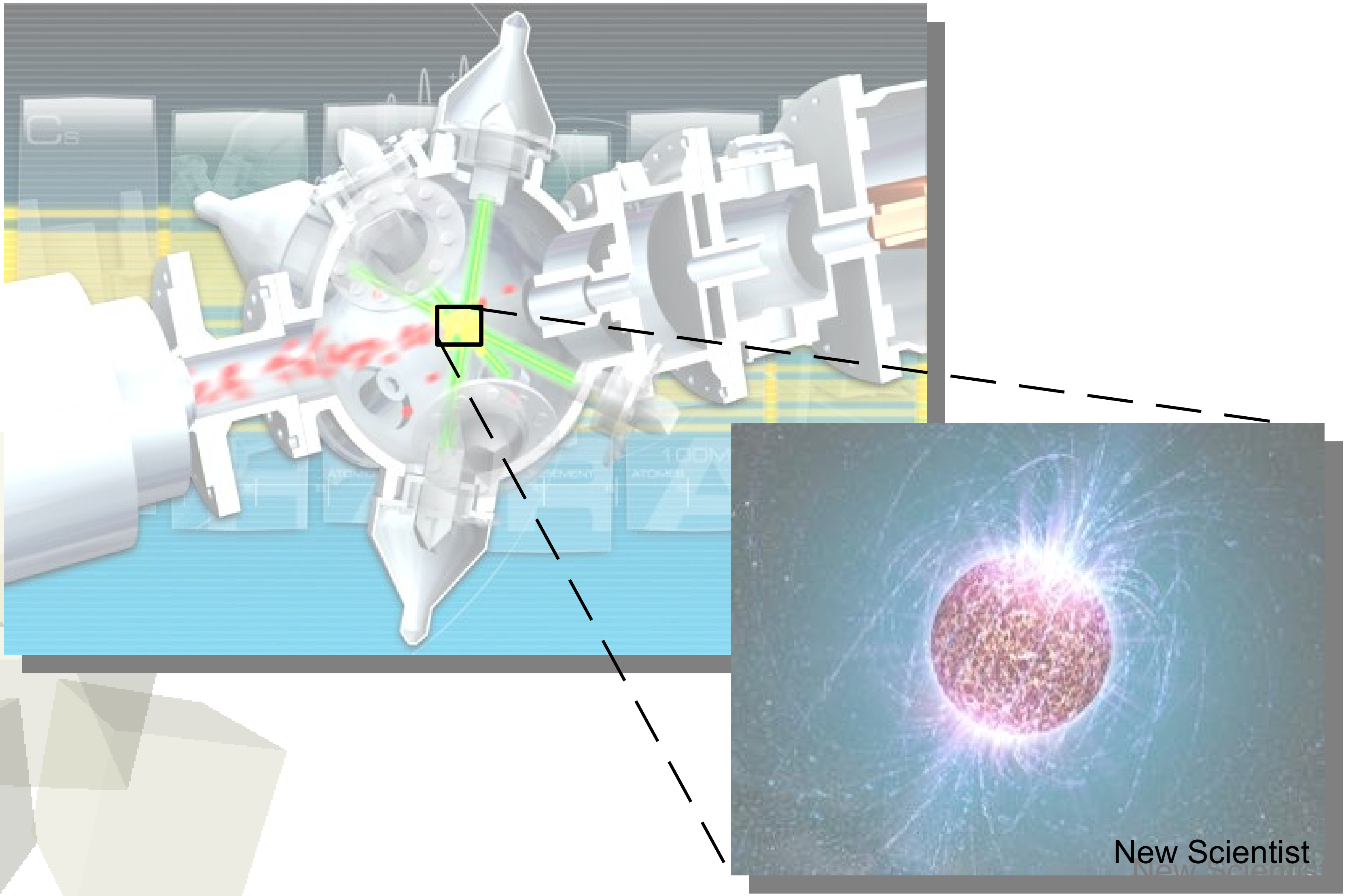
- Experiment

  - Realization of unitary Fermi gases

    e.g., Horikoshi et al., PRX **7**, 041004 (2017)

Quantum simulation of NS matter

Simulating neutron star matter using cold atoms.



Superfluid neutrons in a lattice

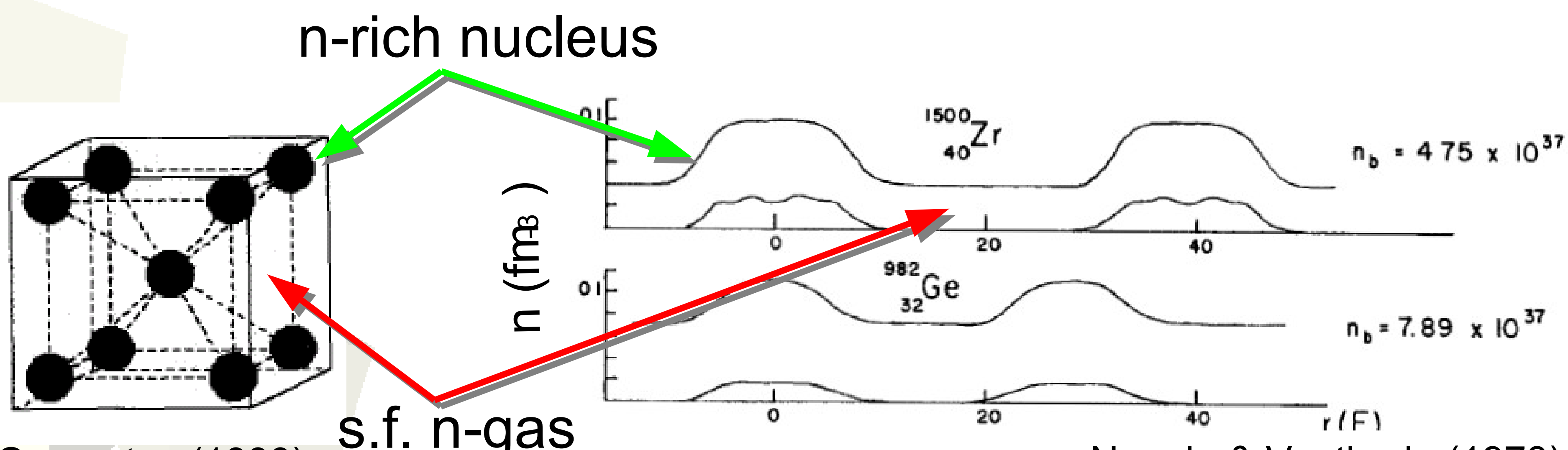
In NS inner crusts

Dripped neutrons btwn. nuclei ≈ s.f. unitary Fermi gas

Lattice of nuclei (normal phase) ≈ periodic pot. for s.f. neutrons

Simulating superfluid neutrons in NS crusts using superfluid Fermi gases in a periodic potential.

e.g., GW, Orso, Dalfovo, Pitaevski & Stringari, PRA **78**, 063619 (2008)
GW, Yoon & Dalfovo, PRL **107**, 270404 (2011)



Oyamatsu (1993)

Negele & Vautherin (1973)

Consequences of superfluidity

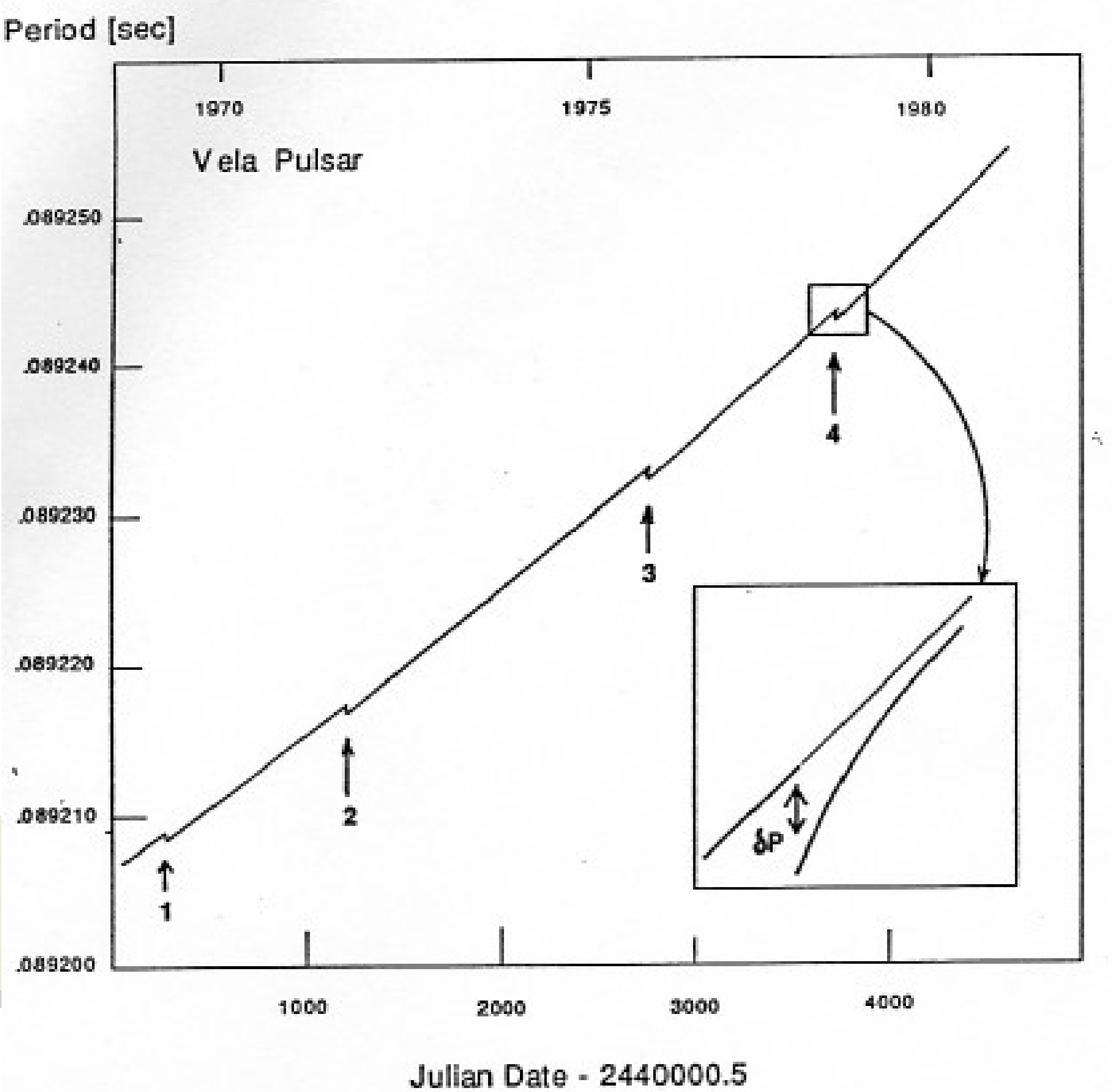
1. Pulsar glitches

Review:
Haskell & Melatos, Int. J. Mod. Phys. D **24**, 1530008 (2015).

2. Neutron star cooling

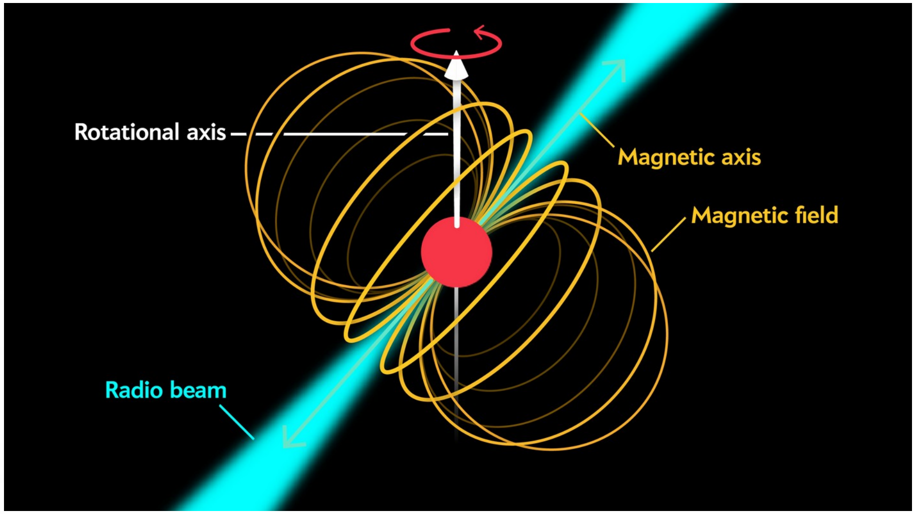
Review:
Yakovlev & Pethick,
Annu. Rev. Astron. Astrophys. **42**, 169 (2004).

Pulsar glitches



Radhakrishnan & Manchester, Nature (1969)

Magnetic dipole model



$$\frac{d\Omega}{dt} \propto -\mu^2 \sin^2\alpha \; \Omega^3 / I$$

μ : magnetic mom.
α : angle btwn. ***B*** & **Ω**
I : mom. of inertia

Glitches: Sudden (but small) increase of pulsar spinning rate.

Followed by a gradual relaxation of the spinning rate.

Glitch model based on n-superfluidity

2 fluids in NS crust

Normal fluid: lattice of nuclei, charged

→ Coupled to an EM field.

Decelerated by pulsar emission.

Superfluid: dripped neutrons

→ No deceleration.

Works as an ang. mom. reservoir.

Relative velocity btwn n-superfluid & lattice of nuclei increases.

Ω_s Ω_n

When the relative vel. goes beyond $\Omega_s - \Omega_n \sim 10 \text{ rad/s}$

→ Sudden ang. mom. transfer from super to normal fluid.

Pulsar glitch!

Superfluid density

How much fraction of neutrons contribute to the superflow is crucial!

Superfluid density n^S

Chamel, PRC 85, 035801 (2012)

GW & Pethick, PRL **119**, 062701 (2017)

sf-hydrodyn.: Martin & Urban, PRC **94**, 065801 (2016)

For slab phase: Kashiwaba & Nakatsukasa, PRC **100**, 035804 (2019)
Sekizawa, Kobayashi & Matsuo, PRC **105**, 045807 (2022)

Disorder effect: Sauls, Chamel & Alpar, arXiv:2001.09959 (2020)
Zhang & Pethick, PRC **105**, 055807 (2022)

Approx. dep.: Minami & GW, PRR **4**, 033141 (2022)

Superfluid density

Superfluid density n^s : Density which contributes to SF flow.

Response of current by the phase twist of the pairing field Δ.

(Increase of the flow kinetic energy by giving a SF velocity.)

energy density

$$n^s_{ij} = m \frac{\partial \mathcal{E}(n, \mathbf{Q})}{\partial Q_i \partial Q_j}$$

$$\Delta(\mathbf{r}) = \tilde{\Delta}(\mathbf{r}) e^{2\mathrm{i}\mathbf{Q}\cdot\mathbf{r}}$$

mom. / particle of bulk superfluid flow

($\propto$ inverse of m^*)

Superfluid density is not condensate density.

$$n^s(T) \neq |\psi|^2$$

(Usually, $n^s(T) \geq |\psi|^2$)

e.g. liquid He-4 @ T=0 : $n^s/n = 1$ but $|\psi|^2/n \sim 0.1$

Entrainment

"Entrainment":

Neutron superfluid is dragged by the lattice of plasma [normal neutron+proton mixture] (and vice versa).

$$\boldsymbol{p}_n = \rho_{nn}\boldsymbol{v}_n + \rho_{np}\boldsymbol{v}_p$$

$$\boldsymbol{p}_p = \rho_{np}\boldsymbol{v}_n + \rho_{pp}\boldsymbol{v}_p$$

ρ_{ij} : entrainment tensor

Entrainment effect means that $\rho_{np} \neq 0$

Usually, entrainment effect reduces the superfluid density of n.

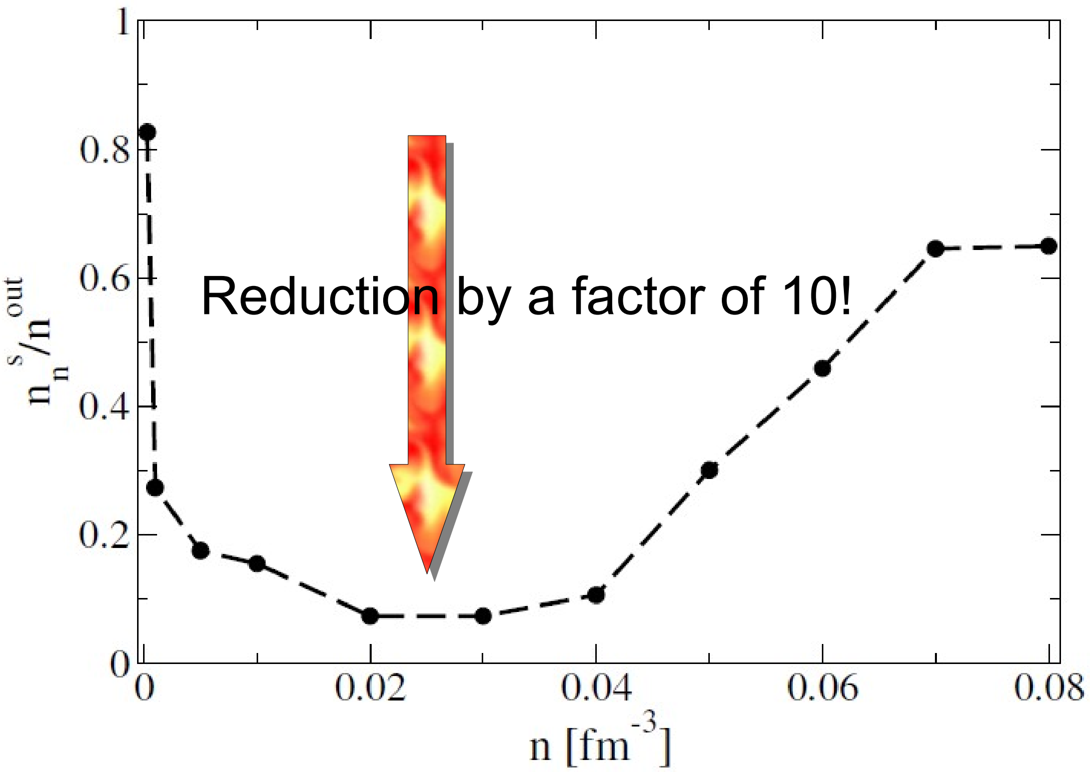
Crisis of the glitch models

Chamel, PRC **85**, 035801 (2012)

Band calculation without pairing.

(HF with nuclear interaction of Skyrme type)

$$(n^S / n \text{ for small but nonzero } \Delta) \approx (m / m^* \text{ for } \Delta = 0)$$

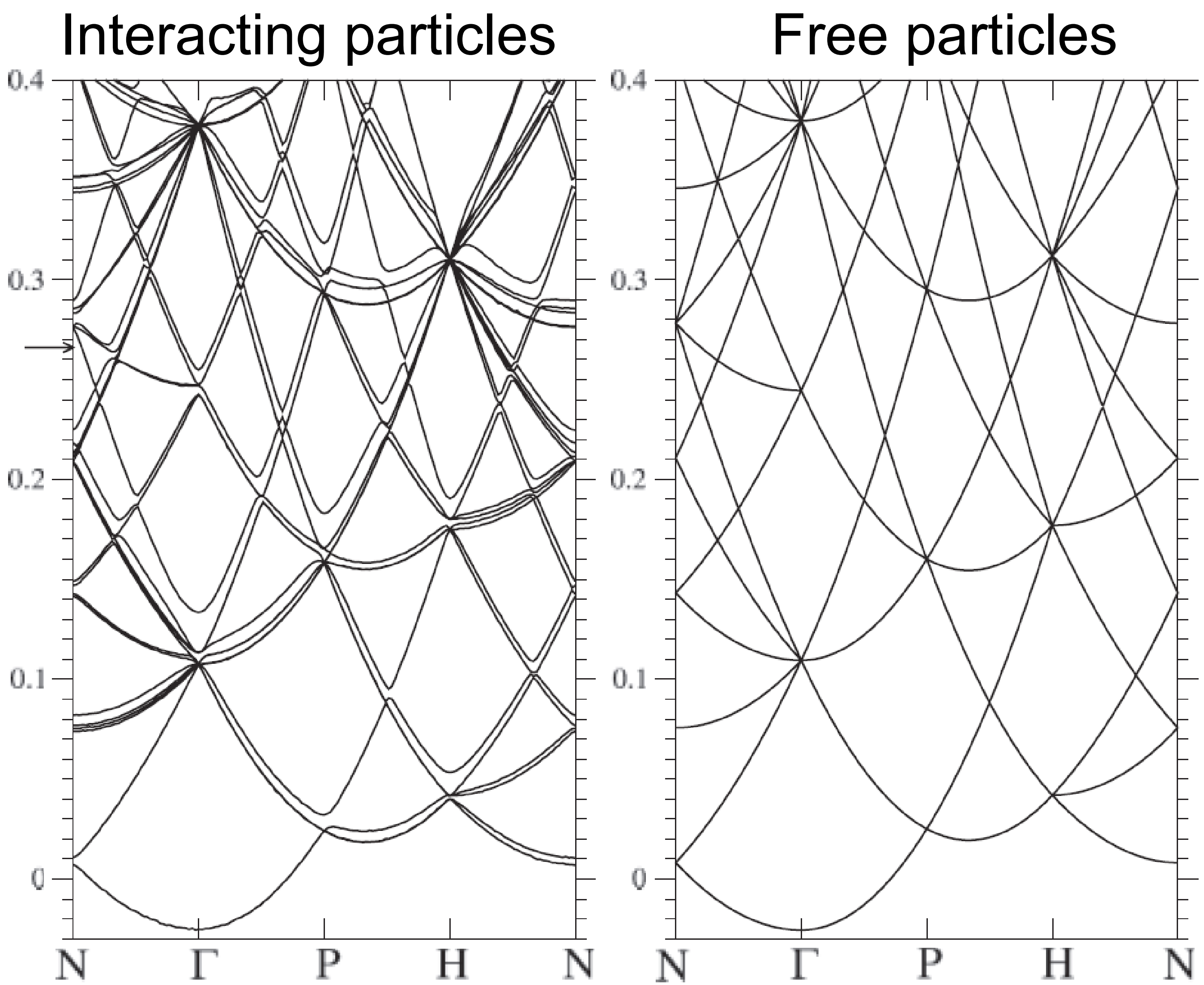


Insufficient superfluid density to explain glitches!

Mom. of inertial of n-superfluid is too small.

Andersson *et al*., PRL (2012); Chamel, PRL (2013); Delsate *et al*., PRD (2016).

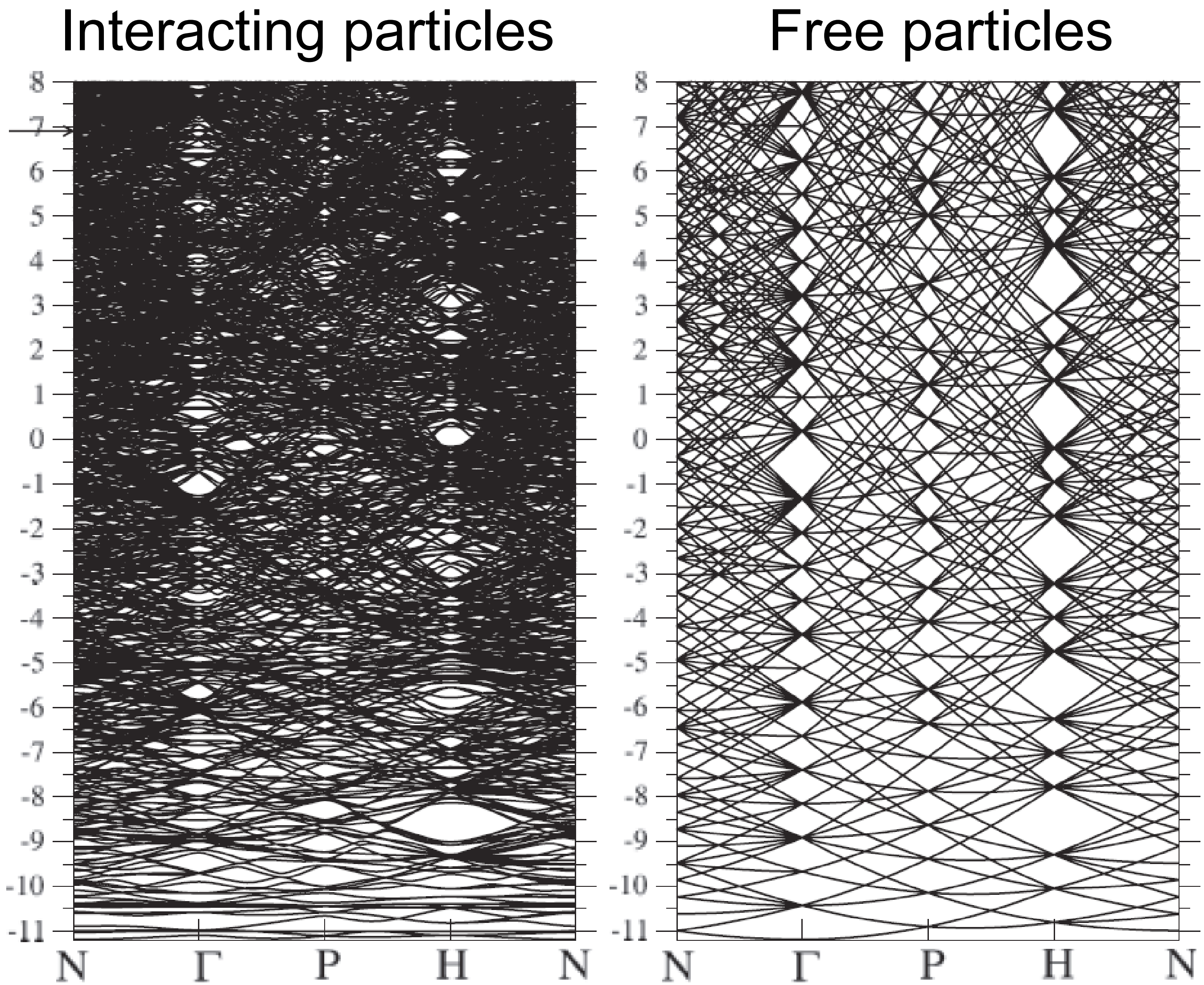
Band calculation by Chamel



$$n = n_n + n_p = 0.0003\ \mathrm{fm}^{-3}$$

Chamel, PRC (2012)

Band calculation by Chamel



$$n = n_n + n_p = 0.03 \text{ fm}^{-3}$$

Chamel, PRC (2012)

dripped n / nucleus is large!

Many avoided crossings near the Fermi surface. → Flat dispersion.

Chamel's updated results by HF+BCS

Chamel (2018; unpublished)

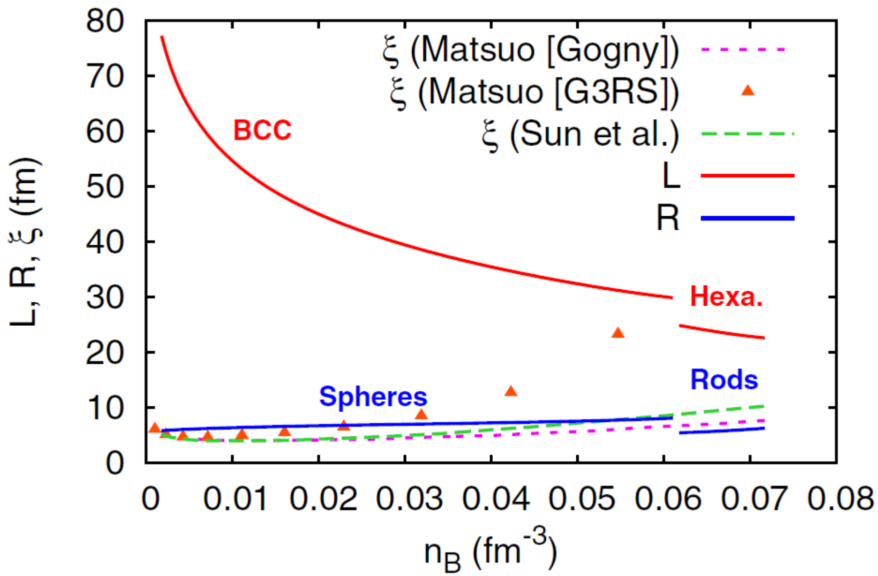
New calculation including pairing gap within HF+BCS

Δ (MeV)	Δ/ε_F	n_n^s/n_n^f (%)
3.09	0.169	7.87
2.16	0.118	7.74
1.51	0.0826	7.63
1.06	0.0578	7.56
0.741	0.0405	7.55
0.519	0.0283	7.57
0.363	0.0198	7.61
0.254	0.0139	7.66
0.178	0.00972	7.77
0.125	0.00680	7.76
0	0	7.84

Chamel, INT Workshop (2018)

Superfluid density is still low!

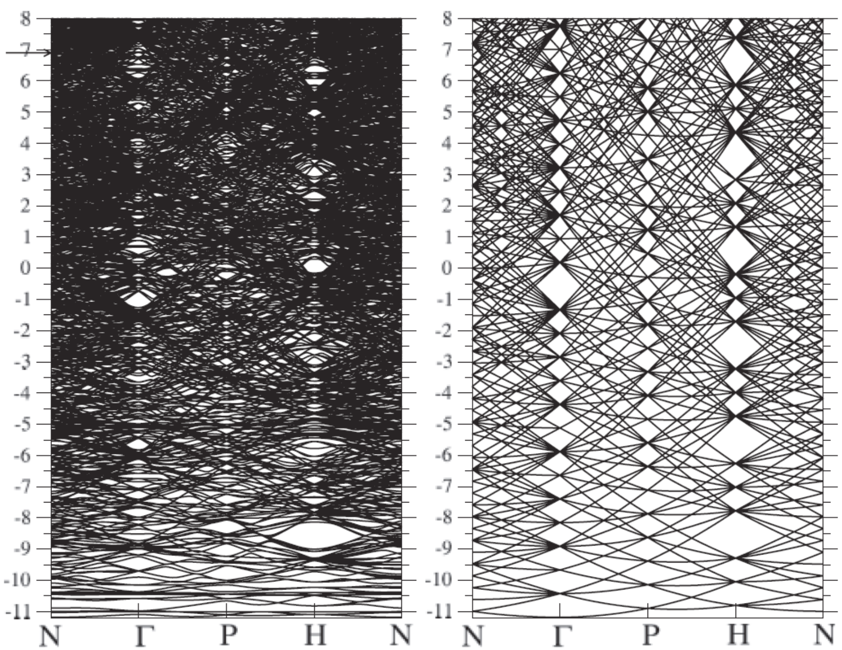
Difficulty of the problem



Martin & Urban, PRC (2016)

$\xi_{\text{BCS}} \sim R$ (nuclear radius)

➡ Hydrodyn. theory is invalid.



Chamel, PRC (2012)

Need to look at the band structure in detail.

# of neutrons / nucleus >> 1

neutrons occupy ~500 bands

Take-home messages

Both "gaps" are important!

(band gap) vs (pairing gap)

- Pairing gap and band gap should be treated on equal-footing!

- Pairing drastically reduces the effects of band gap when

$$|\Delta| \gtrsim \text{(band gap)} \sim \text{(inter-band coupling)}$$

$|\Delta|/(\text{band gap})$ matters even though $|\Delta|/E_F \ll 1$

- Superfluid density may be large enough to account for glitches.

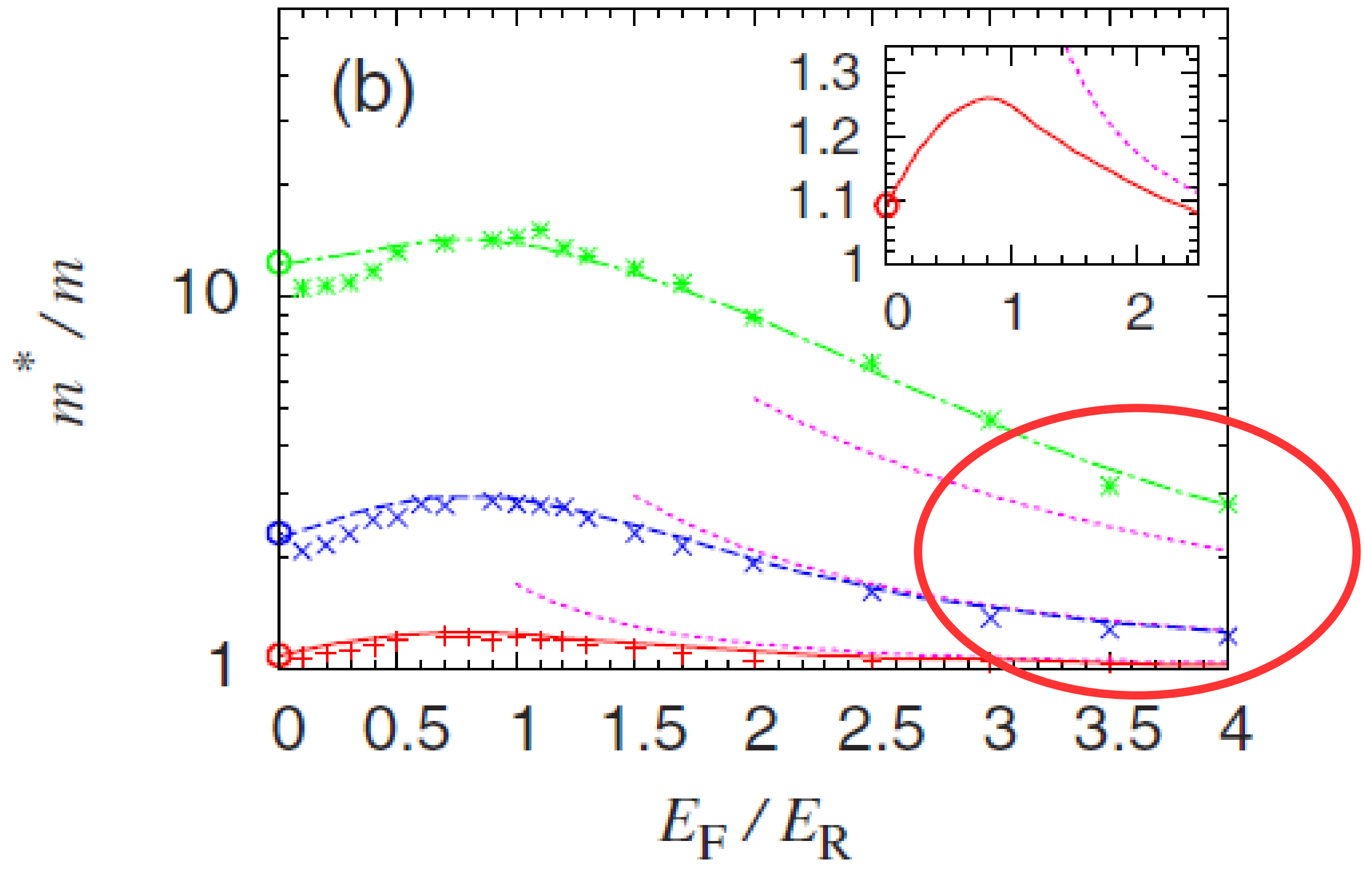
New life for glitch models!

GW & Pethick, PRL **119**, 062701 (2017).

Minami & GW, PRR **4**, 033141 (2022).

Our old result on superfluid Fermi gases in an optical lattice

GW *et al*. PRA **78**, 063619 (2008)



Effective mass is not so big at higher densities...

Pairing suppresses the effects of band gap...?

Poor man's analysis

Scattering of quasiparticles by spin-indep. pot.:

$$H_{\rm int} = \sum_{\mathbf{k},\mathbf{k}',\sigma} V(|\mathbf{k}-\mathbf{k}'|) a^\dagger_{\mathbf{k},\sigma} a_{\mathbf{k}',\sigma}$$

$$a^\dagger_{\mathbf{k},\sigma} = u_k \alpha^\dagger_{\mathbf{k},\sigma} + \sigma v_k \alpha_{-\mathbf{k},-\sigma}$$

fermion — $a^\dagger_{\mathbf{k},\sigma}$; quasiparticle — $\alpha^\dagger_{\mathbf{k},\sigma}$, $\alpha_{-\mathbf{k},-\sigma}$

$$\langle \mathbf{k}'\sigma | H_{\rm int} | \mathbf{k}\sigma \rangle = \underline{(u_k u_{k'} - v_k v_{k'})} V(|\mathbf{k}-\mathbf{k}'|)$$

$$|\mathbf{k}\sigma\rangle = \alpha^\dagger_{\mathbf{k},\sigma}|0\rangle$$

On the Fermi surface ($k = [2m\mu]^{1/2}$), $u_k = v_k = 1/\sqrt{2}$

$$\Rightarrow \langle \mathbf{k}'\sigma | H_{\rm int} | \mathbf{k}\sigma \rangle = 0$$

No net scattering on Fermi surface.

∵ Potential for particles and holes are opposite in sign.

Poor man's analysis

Potential "mountain" for particles is potential "valley" for holes.

Pairing ➡ Superposition of particle & hole.

Quasi-particles are insensitive to the potential on average.

Pairing suppresses the effect of the band gap.

Bogoliubov-de Gennes approach

Basic equation (BdG eq.):

energy of st. with $\mathbf{k}$

Amp. of $\mathbf{k}\uparrow$ & $-\mathbf{k}\downarrow$ are empty

$$\begin{pmatrix} \tilde{H}'_{\mathbf{Q}}(\mathbf{r}) & \tilde{\Delta}(\mathbf{r}) \\ \tilde{\Delta}^*(\mathbf{r}) & -\tilde{H}'_{-\mathbf{Q}}(\mathbf{r}) \end{pmatrix} \begin{pmatrix} \tilde{u}_i(\mathbf{r}) \\ \tilde{v}_i(\mathbf{r}) \end{pmatrix} = \epsilon_i \begin{pmatrix} \tilde{u}_i(\mathbf{r}) \\ \tilde{v}_i(\mathbf{r}) \end{pmatrix}$$

Amp. of $\mathbf{k}\uparrow$ & $-\mathbf{k}\downarrow$ are occupied

$\mathbf{Q}$: quasimom. per particle of superflow

$$\tilde{H}'_{\mathbf{Q}}(\mathbf{r}) = \frac{1}{2m}(-\mathrm{i}\boldsymbol{\nabla} + \mathbf{Q} + \mathbf{k})^2 + V_{\mathrm{ext}}(\mathbf{r}) - \mu$$

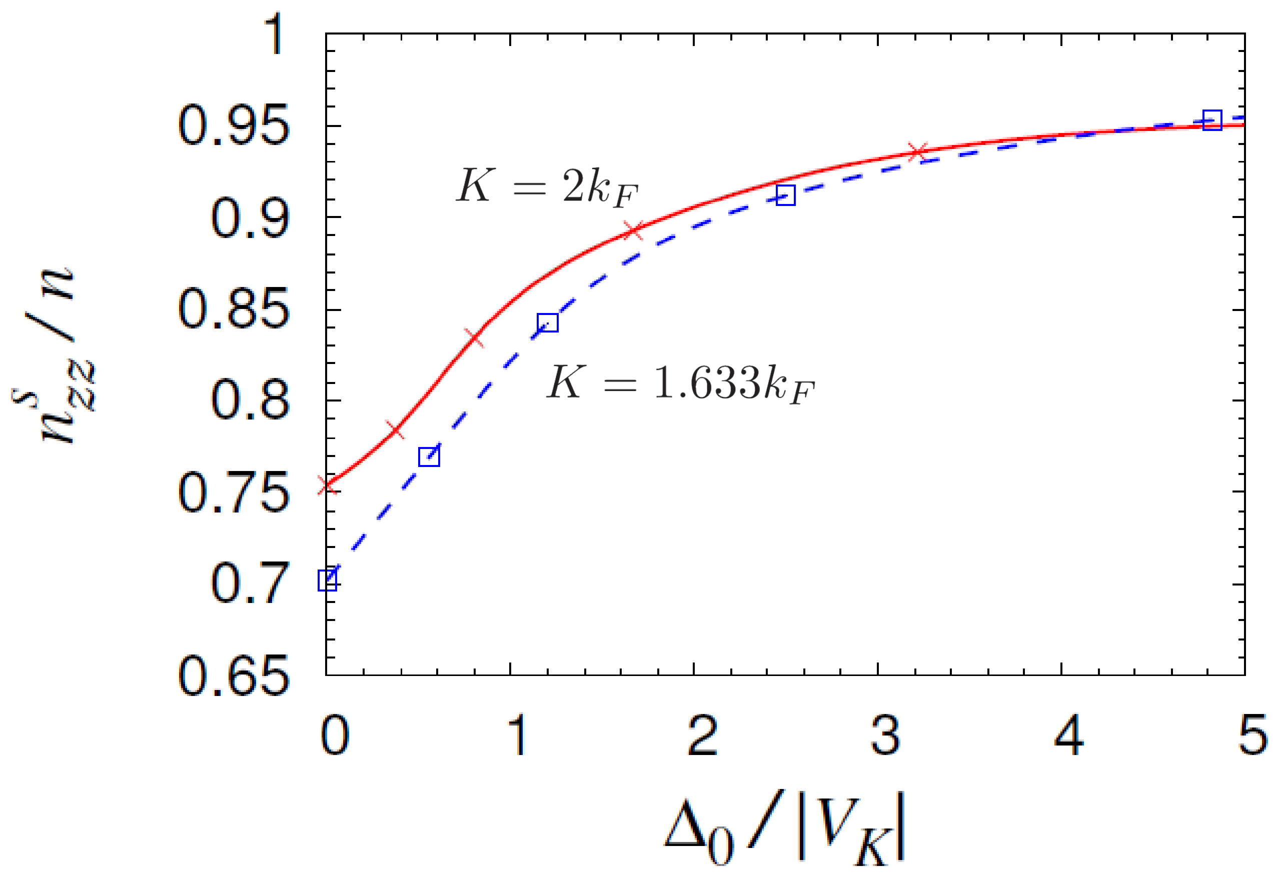
1D sinusoidal pot.

$$V_{\mathrm{ext}}(\mathbf{r}) = V_K(e^{\mathrm{i}Kz} + e^{-\mathrm{i}Kz})$$

Calculate $$n^s_{ij} = m\frac{\partial \mathcal{E}(n, \mathbf{Q})}{\partial Q_i \partial Q_j}$$

Effects of the pairing gap

Suppression of band gap effect by pairing



Approximate fit: $$n^s_{zz}(\Delta_0) = n - \frac{n - n^s_{zz}(0)}{\sqrt{1 + (\Delta_0/|V_K|)^2}}$$

suppression factor

Application to NS crusts

Obstacles:

- Lattice pot. in NS crusts has many Fourier components.
- 3D anisotropic lattice:
  Average over the orientation of lattice is needed.
- Neutrons occupy ~ 500 bands.

→ Direct BdG approach is formidable.

Take a shortcut!

Use the results for cold atom system.

Application to NS crusts

Shortcuts:

- Lattice pot. in NS crusts has many Fourier components.

Not simple sinusoidal

→ Form factor

(from MF pot. used in Chamel's calc.)

[Pearson *et al*., PRC **91**, 018801 (2015)]

- 3D bcc lattice

Superfluid density tensor

→ Average over the orientation of lattice.

- K-dep. & Δ-dep.: BdG results for the simplified sys.

Effects of the two "gaps".

Superfluid density in NS crusts

Assumption: pairs of RLVs {$\mathbf{K}_i$, -$\mathbf{K}_i$} contribute to n^s independently.

$$n^s_{ij}(\mathbf{K}) = n^s_{zz}\hat{K}_i\hat{K}_j + n[\delta_{ij} - \hat{K}_i\hat{K}_j]$$

longitudinal transverse

orientation avr. & sum over RLV pairs

sum → integral

$$\frac{n^s}{n^o_n} \approx \exp\left\{-\frac{2n^o_n}{n_N}\int_0^1 x^2 dx \left[1 - \frac{n^s_{zz}(K, V_K, \Delta)}{n}\right]\right\}$$

effect of lattice pot.

$$1 - \frac{n^s_{zz}(K, V_K, \Delta)}{n} \approx \left[1 - \frac{n^s(K, V_K, \Delta = 0)}{n}\right] \frac{1}{\sqrt{1 + (\Delta/|V_K|)^2}}$$

$$\approx (1 + 3.5x)\frac{|V_K|}{E_F}$$

$x \equiv K/2k_n$

K-dep. (from BdG)

approximate fit

form factor of lattice pot. (V_K-dep.)

effect of pairing gap (from BdG)

Superfluid density in NS crusts

Focus on the case where the reduction of n^S is largest.

In Chamel (2012): Avr. density n = 0.03 fm^{-3}

neutron Fermi energy $E_F{}^o$ = 16.4 MeV

$$\Delta \approx 1 - 1.5\ \mathrm{MeV}$$

No pairing limit (Δ = 0): $n^s/n_n^o \simeq 0.20$ (cf. Chamel's result ~ 0.1)

Δ = 1MeV: $n^s/n_n^o \simeq 0.64$

Δ = 1.5MeV: $n^s/n_n^o \simeq 0.71$ Only 29% reduction!

Superfluid density is large enough.

→ Glitch models based on superfluidity are still tenable!

Comparison

- GW & Pethick, PRL (2017)

  BdG equation for 1D lattice in 3D space

  Band gap & paring gap: included on equal-footing

- Chamel, PRC (2012); unpublished (2018)

  HF (no pairing) for 3D lattice (2012)

  HF-BCS for 3D lattice (2018)

  Include band gap → paring gap

We focus on the dependence of approximation scheme.

Minami & GW, PRR **4**, 033141 (2022)

Minimal model

1D periodic potential

$$V_{\text{ext}}(x) = V\left(e^{iKx} + e^{-iKx}\right)$$

Bloch states with 3 bands

$$u_k(x) = \sum_{n=0,\pm1} \tilde{u}_{k+nK} e^{i(k+nK)x}$$

BdG eqs. in 1D space

$$\begin{pmatrix} \tilde{H}'(k,Q) & \tilde{\Delta} \\ \tilde{\Delta} & -\tilde{H}'(k,-Q) \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{u}} \\ \tilde{\boldsymbol{v}} \end{pmatrix} = \epsilon \begin{pmatrix} \tilde{\boldsymbol{u}} \\ \tilde{\boldsymbol{v}} \end{pmatrix}$$

Three approximation schemes

(1) Directly solve BdG eqs. and calculate superfluid density: n^s_{BdG}

V & Δ are on equal footing.

(2) First include V for normal state, and then include Δ: $n^s_{V\Delta}$

(Corresponding to HF-BCS approx.)

First, diagonalize

$$\tilde{H}'(k,Q) = \begin{pmatrix} \xi(k+K,Q) & V & 0 \\ V & \xi(k,Q) & V \\ 0 & V & \xi(k-K,Q) \end{pmatrix}, \quad \xi(k,Q) = \frac{1}{2m}(k+Q)^2 - \mu(Q)$$

→ Then, include p-h mixing for $\tilde{H}'(k,Q)$ basis.

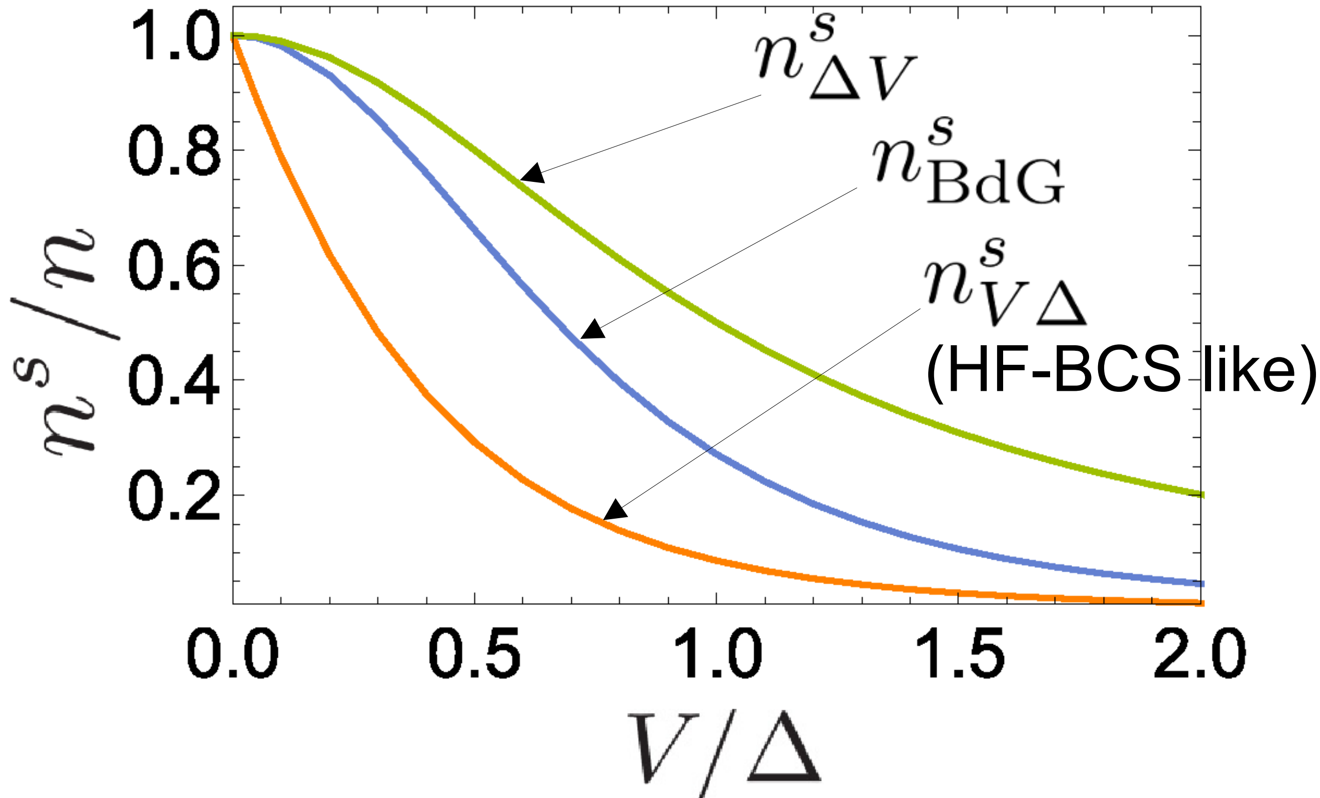
(3) First include Δ for homogeneous sys., and then include V: $n^s_{\Delta V}$

First, solve BCS for homogeneous sys.

→ Then, include band mixing by V.

Superfluid densities by the 3 methods

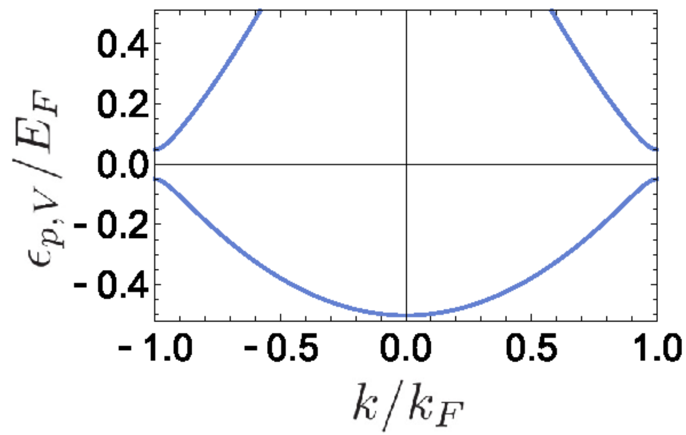
Even at $\Delta \ll E_F$, n^S differs significantly when $V \sim \Delta$.



Δ / E_F = 0.01

K / k_F = 2: nested

➡ Fermi points are inside the band gap.



Minami & GW, PRR (2022)

$n^S_{\Delta V}$ > n^S_{BdG} > $n^S_{V\Delta}$

(equal-footing) (HF-BCS like)

Ratio to the BdG result

For $\Delta/E_F = 0.01$

Minimum:

$$n^s_{V\Delta}/n^s_{\mathrm{BdG}} \sim 0.3$$

HF-BCS like

Maximum:

$$n^s_{\Delta V}/n^s_{\mathrm{BdG}} \sim 8$$

V and Δ should be treated on equal footing in the region of $\Delta \sim V$ even if $\Delta << E_F$.

Discussions

• HF-BCS approx. can considerably underestimate n^S in NS crusts where $\Delta \gtrsim V$.

$\Delta \sim$ 1–2 MeV $\gtrsim V$ in inner crusts

• Nested in 1D: Most “conservative” case.

Reduction of n^S by V is most prominent.

At $\Delta = V$: $n^S{}_{\mathrm{BdG}} / n = 0.27$ ($n^S{}_{V\Delta} / n = 0.089$)

n^S in NS crusts (bcc lattice in 3D) would be larger.

Summary & conclusion

n^S is determined by subtle balance btwn. the two gaps!

- Both pairing gap and band gap are important.

  $|\Delta|/(\text{band gap})$ matters rather than $|\Delta|/E_F$

- Approximation scheme matters when $\Delta \sim V$ even if $\Delta << E_F$.

  Δ and V should be treated on equal footing for NS crusts!

- Effects of the band gap is suppressed in NS crusts.

  $n^s/n_n^o \sim 0.1$ → $\simeq 0.7$

  No pairing → Pairing included

Pulsar glitch models get new life!

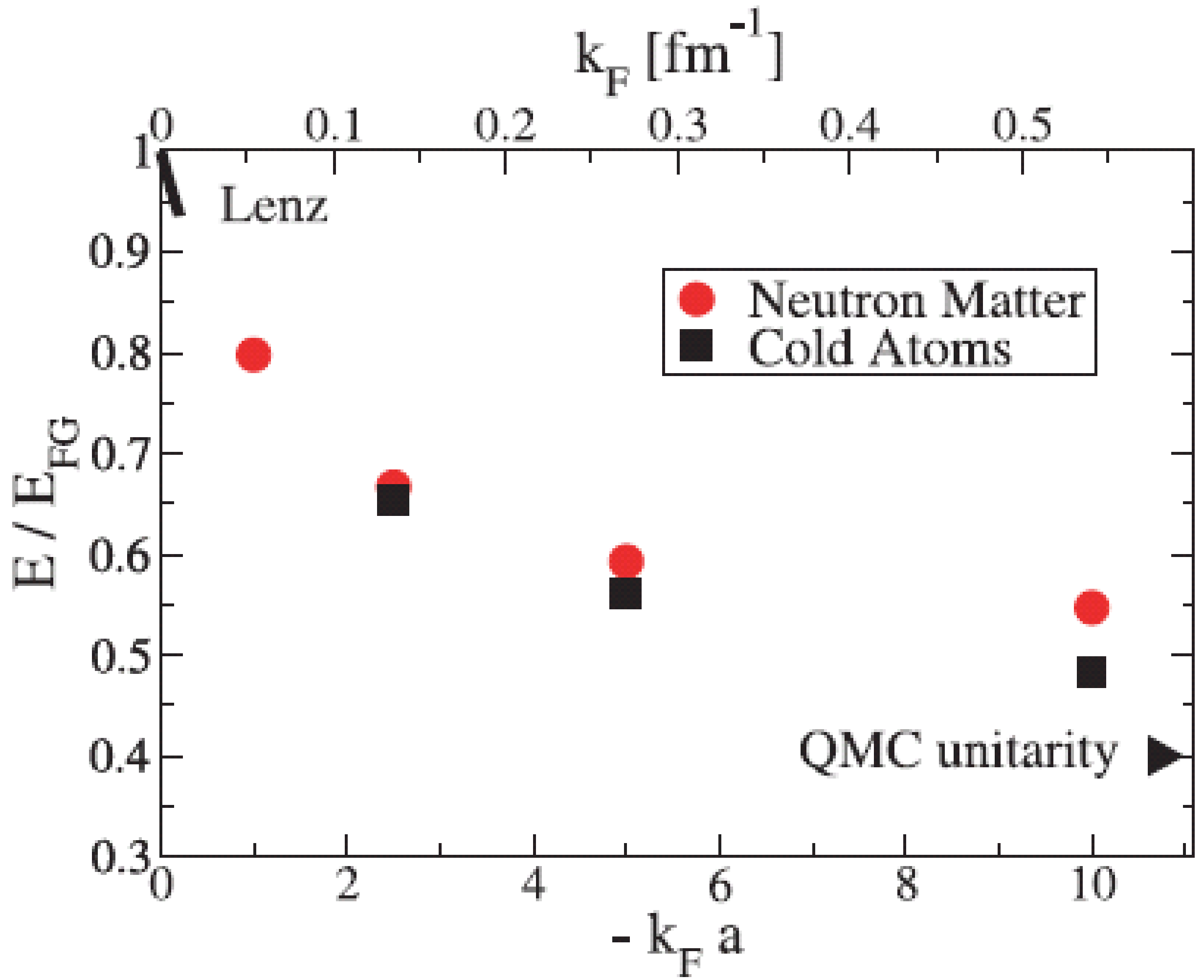
GW & Pethick, PRL **119**, 062701 (2017).
Minami & GW, PRR **4**, 033141 (2022).

Comparison btwn n-matter & cold atomic gases

		Neutron matter in NS crust $(\rho \lesssim \rho_0)$	Cold Fermi gas at unitarity
Particle separation	$r_s \sim k_F^{-1}$	~ 1 fm	~ 100 nm
Scattering length	a_s	-18.9 fm	∞
	$k_F \lvert a_s \rvert$	~ 19 >> 1 very large!	∞
Temperature	T	~ 100 keV	~ 100 nK
Degeneracy temp.	T_F	~ 100 MeV	~ 1 μK
	T/T_F	~ 10^{-3}	~ 0.1

QMC results for T=0

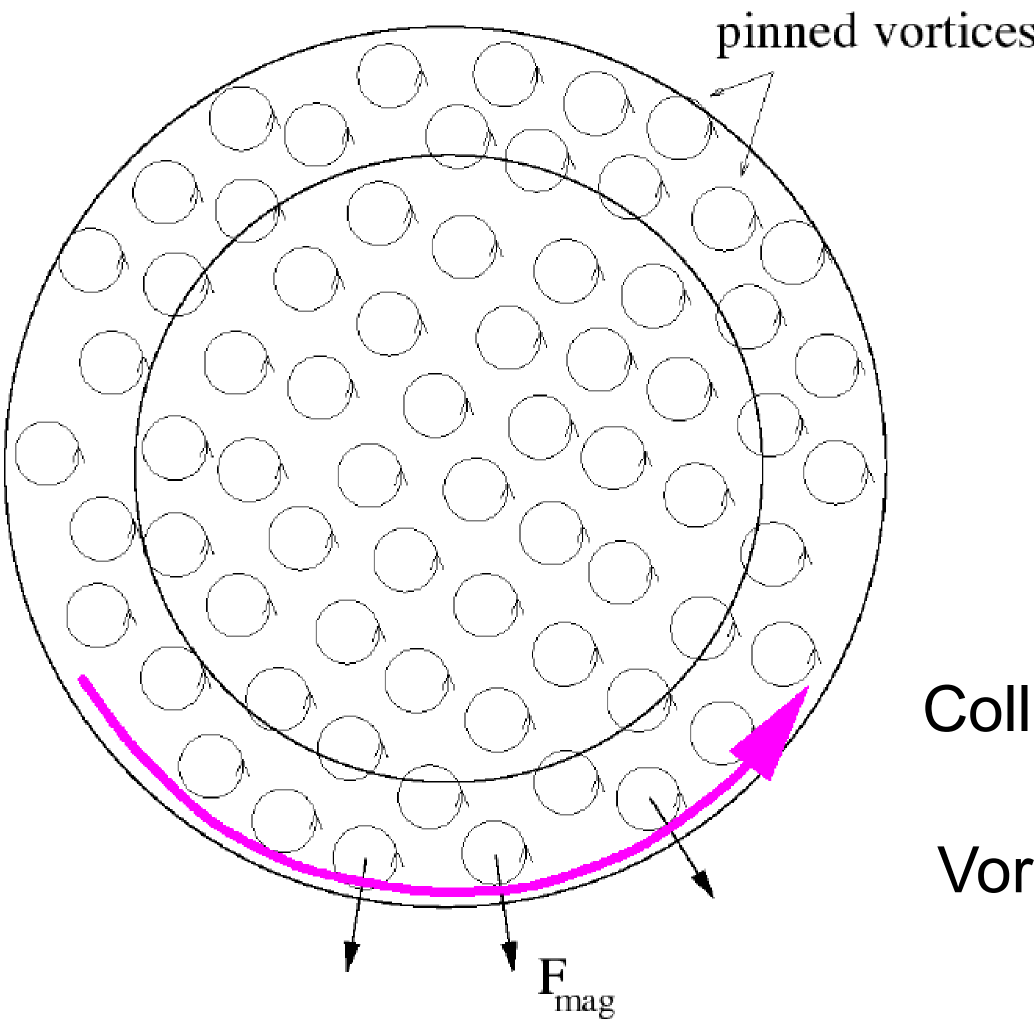
Quantum Monte-Carlo by Gezerlis & Carlson (2008)



Gezerlis & Carlson, PRC **77**, 032801(R) (2008)

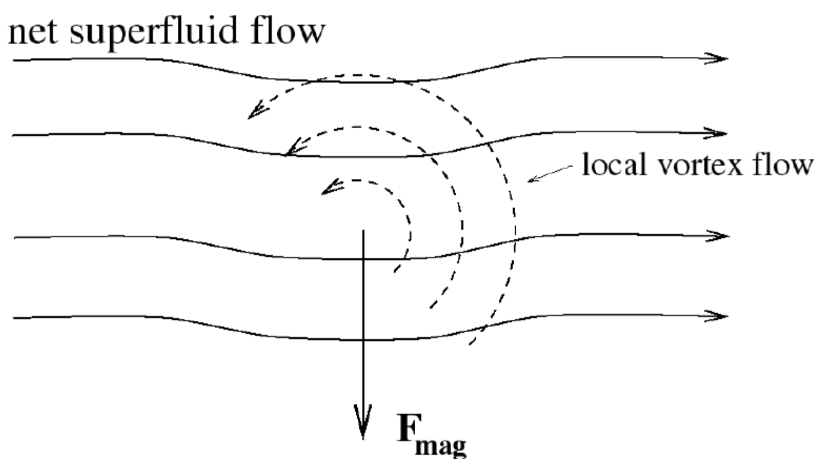
Good agreement btwn n-matter & cold atoms.

Glitch model based on n-superfluidity



Magnus force

$$\boldsymbol{F}_{\rm mag} = -\rho_s \boldsymbol{\kappa} \times (\boldsymbol{v}_s - \boldsymbol{v}_{\rm vortex})$$



Collective unpinning

Vortex density $n_v \downarrow$

$\oint \boldsymbol{v}_s \cdot d\boldsymbol{l} = \frac{h}{m} N_v$ ⇒ $\Omega_s = \pi \frac{\hbar}{m} n_v$

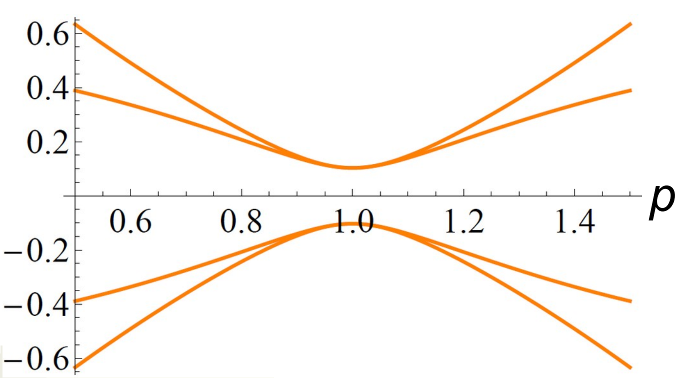
$\Omega_s \downarrow$ ⇒ $\Omega_n \uparrow$

ang. mom. transfer

Simple analysis by 2-band model

particle (p)	hole (p)	particle (p-K)	hole (p-K)

$$H = \begin{pmatrix} \frac{p^2}{2} - \frac{1}{2} & \Delta & V & 0 \\ \Delta & -\left(\frac{p^2}{2} - \frac{1}{2}\right) & 0 & -V \\ V & 0 & \frac{1}{2}(p-K)^2 - \frac{1}{2} & \Delta \\ 0 & -V & \Delta & -\left[\frac{1}{2}(p-K)^2 - \frac{1}{2}\right] \end{pmatrix}$$



p : quasimom. of a quasiparticle (in units of p_F)

K : reciprocal lattice vector (in units of p_F)

V : strength of the lattice pot. (in units of $2E_F$)

Δ : pairing gap (in units of $2E_F$)

Nested case: $K = 2$

➡ Eigenvalues @ $p=p_F$: $\pm\sqrt{\Delta^2 + V^2}$ (doubly degenerate)

$|\Delta|/|V| \gtrsim 1$ ➡ Pairing effect is important even if $|\Delta|/E_F \ll 1$

$$E = \int d\mathbf{r} \left[\frac{\hbar^2}{2m} \left(2 \sum_i |\nabla v_i(\mathbf{r})|^2 \right) + V_{\text{ext}}(\mathbf{r}) n(\mathbf{r}) + \frac{1}{g} |\Delta(\mathbf{r})|^2 \right].$$

Standard regularization

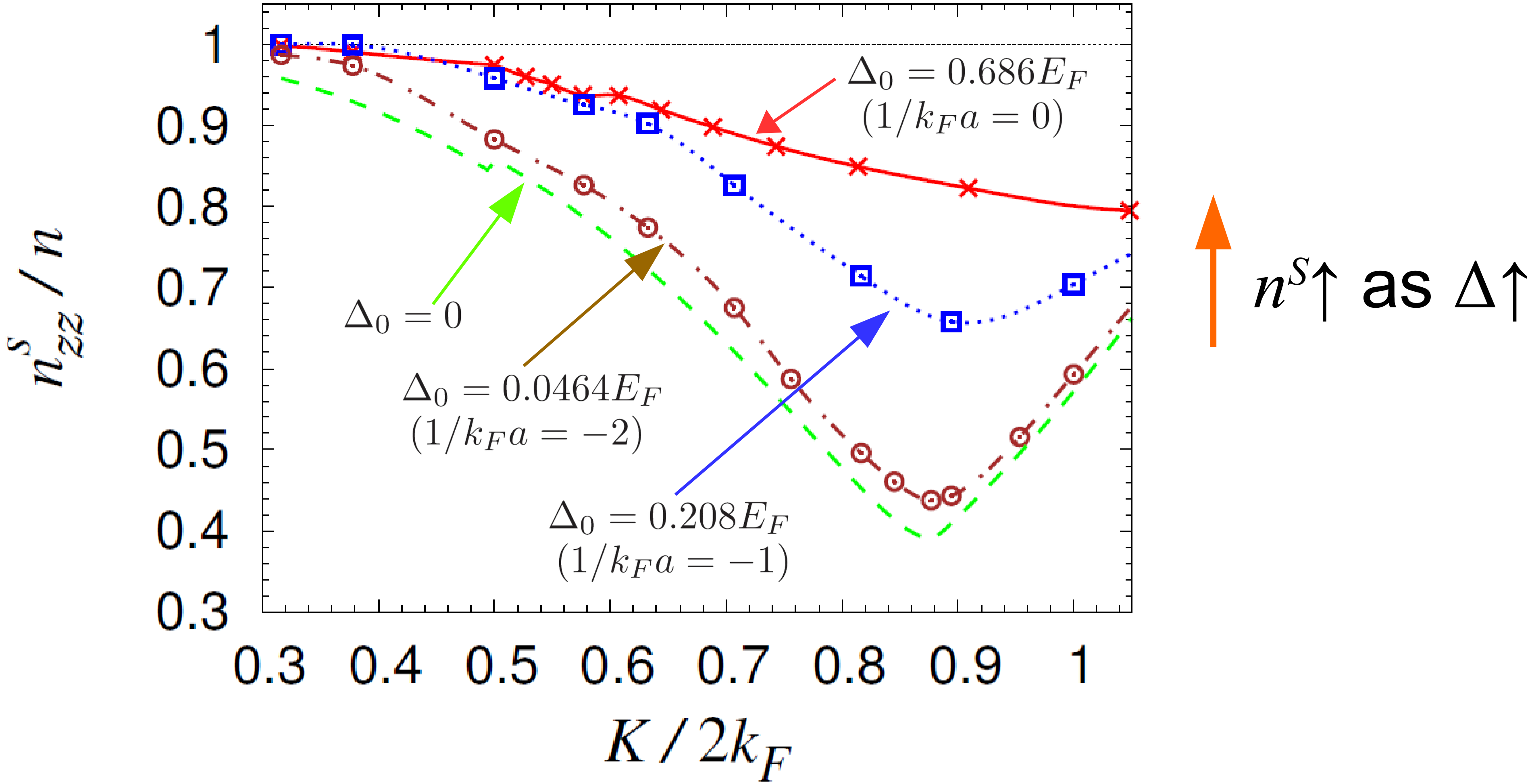
Take contact potential: $g\delta(\mathbf{r})$

Replace *g* by the low energy limit of T-matrix.

$$\frac{1}{g} = \frac{m}{4\pi\hbar^2 a_s} - \sum_{k<k_c} \frac{1}{2\epsilon_k}$$

$$\epsilon_k \equiv \frac{\hbar^2 k^2}{2m}$$

Effects of the pairing gap (1)

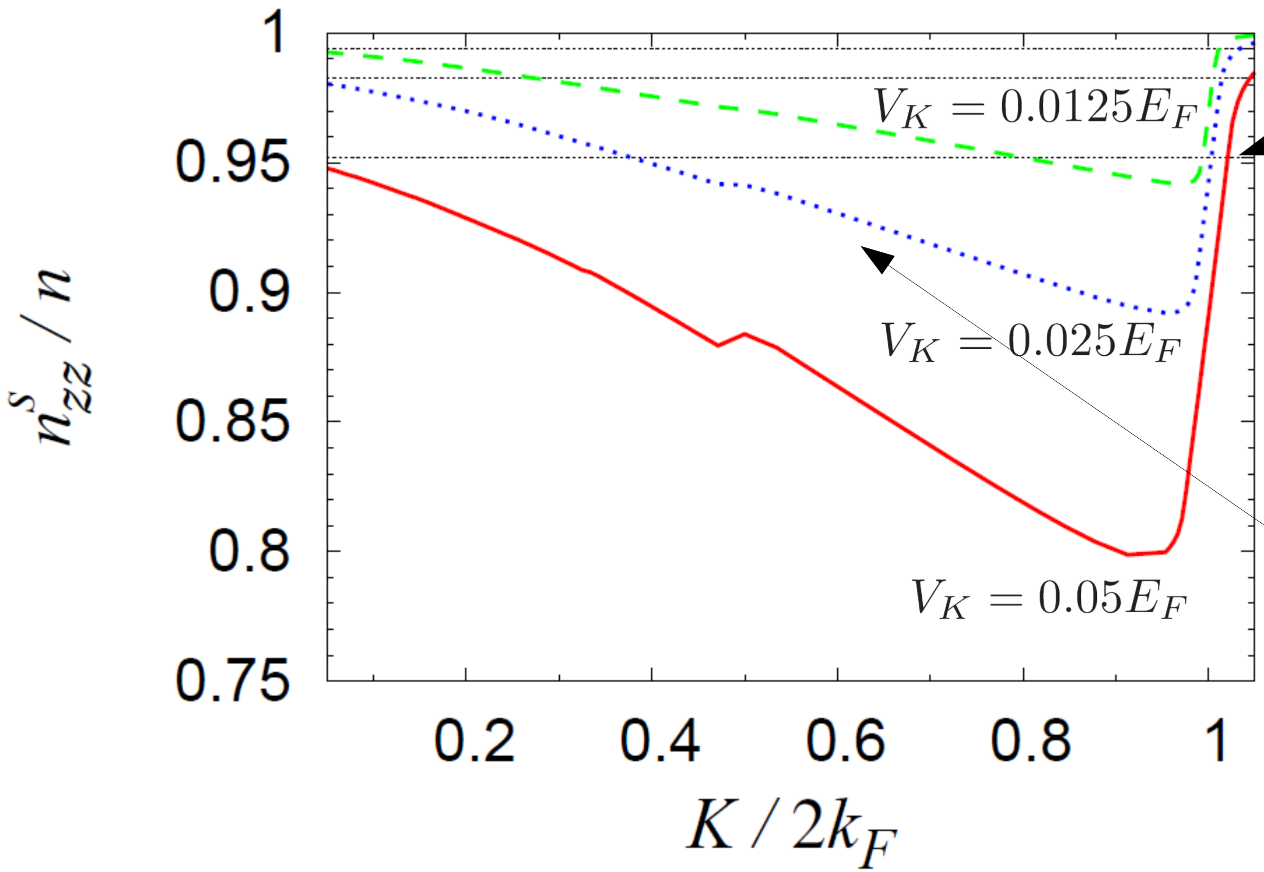


$$V_K = 0.25(K/2k_F)^2 E_F$$

Reduction of n^s due to band gap is suppressed by paring gap.

K and V_K dependence in normal limit

Normal limit: $\Delta = 0$



Little effect of lattice for $K > 2k_F$

Scattering btwn two st. near the Fermi surface must be $K < 2k_F$

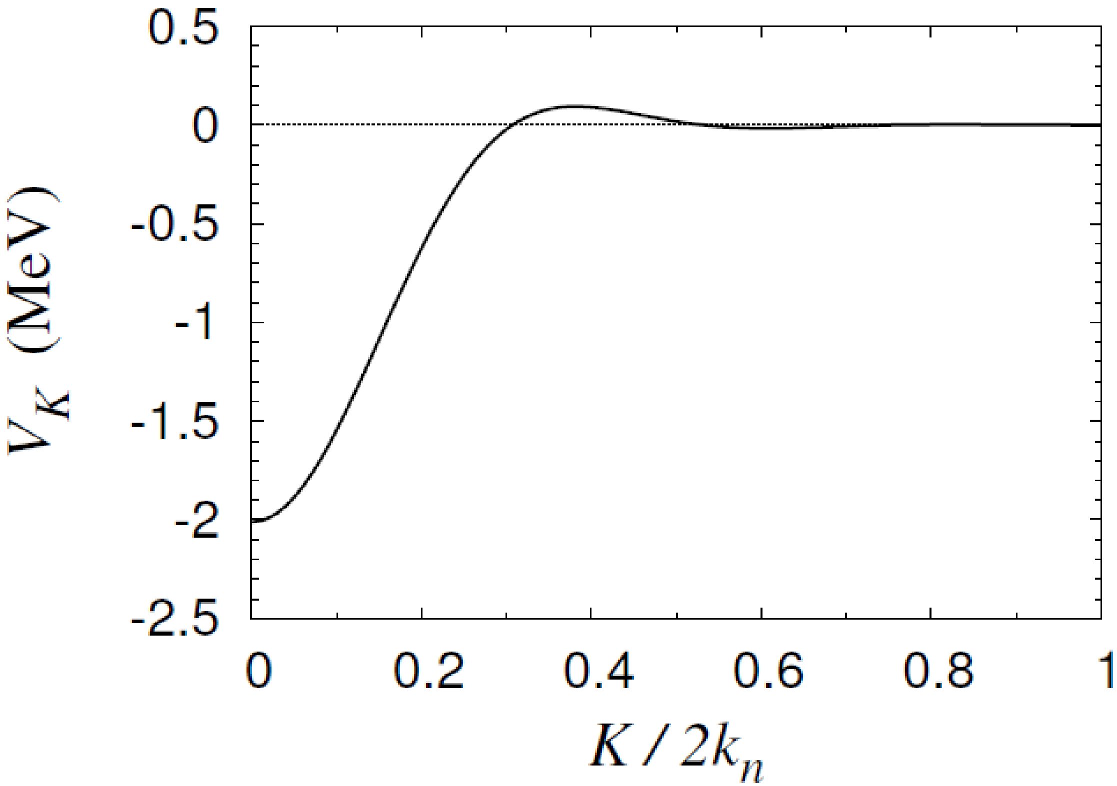
Almost linear wrt $K/2k_F$

Approximate fit: $1 - n^s_{zz}(K, V_K, \Delta = 0)/n = \left(1 + 3.5 \frac{K}{2k_F}\right) \frac{|V_K|}{E_F}$

Form factor of lattice pot. in NS crusts

Fourier transform of MF pot. in Chamel's calculation.

[Pearson *et al.*, PRC **91**, 018801 (2015)]



Reciprocal lattice vectors (RLVs)

bcc lattice → fcc in reciprocal space

Min: $\mathbf{K} = \frac{4\pi}{d}(\pm1, \pm1, 0)$ etc. (12 RLVs)

$$K_{\mathrm{min}}/2k_n \simeq 0.12$$

2nd: $\mathbf{K} = \frac{4\pi}{d}(\pm2, 0, 0)$ etc. (6 RLVs)

$$K = \sqrt{2}K_{\mathrm{min}}$$

3rd: $\mathbf{K} = \frac{4\pi}{d}(\pm1, \pm1, \pm2)$ etc. (24 RLVs)

$$K = \sqrt{3}K_{\mathrm{min}}$$

$|V_K|$ decreases rapidly with K.

$K/2k_n \gtrsim 0.15$ ➡ $|V_K| \lesssim \Delta$

$K/2k_n \gtrsim 0.25$ ➡ $|V_K| \ll \Delta$

Superfluid density in NS crusts (1)

Assumption: pairs of RLVs {**K**$_i$, -**K**$_i$} contribute to n^s independently.

$$n^s_{ij}(\mathbf{K}) = n^s_{zz}\hat{K}_i\hat{K}_j + n[\delta_{ij} - \hat{K}_i\hat{K}_j]$$

longitudinal transverse

avr. over orientation for cubic symmetry

$$\frac{n^s}{n} = 1 - \frac{1}{3}[1 - n^s_{zz}(K, V_K, \Delta)/n]$$

contribution from many RLVs in crusts

Only one of **K**$_i$ & -**K**$_i$ is included.

$$\frac{n^s}{n^o_n} = \prod_{\mathbf{K}_i}{}' \left\{1 - \frac{1}{3}\left[1 - \frac{n^s_{zz}(K, V_K, \Delta)}{n}\right]\right\}$$

$$\approx \exp\left\{-\frac{1}{6}\sum_{\mathbf{K}_i}\left[1 - \frac{n^s_{zz}(K, V_K, \Delta)}{n}\right]\right\}$$

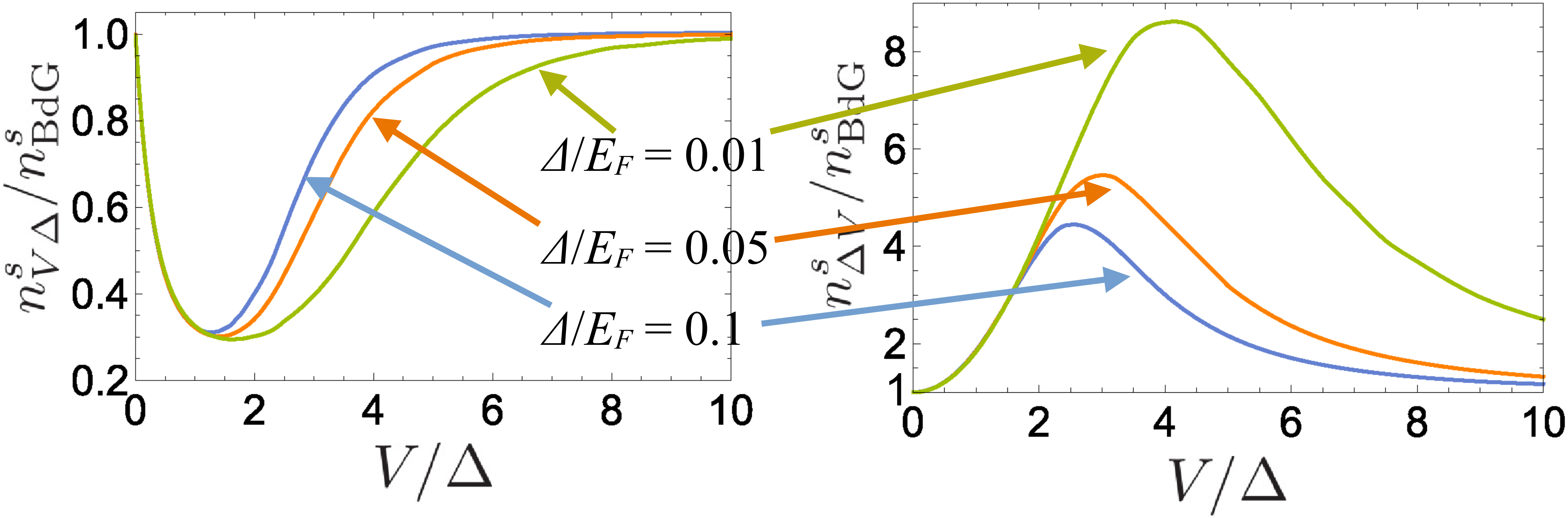
sum → integral

$$\frac{n^s}{n^o_n} \approx \exp\left\{-\frac{2n^o_n}{n_N}\int_0^1 x^2 dx \left[1 - \frac{n^s_{zz}(K, V_K, \Delta)}{n}\right]\right\}$$

density of nuclei $x \equiv K/2k_n$

Ratio to the BdG result

HF-BCS like



Minami & GW, PRR (2022).

For $\Delta/E_F = 0.01$

Minimum:

$$n^s_{V\Delta}/n^s_{\rm BdG} \sim 0.3$$

Maximum:

$$n^s_{\Delta V}/n^s_{\rm BdG} \sim 8$$

V and Δ should be treated on equal footing in the region of $\Delta \sim V$ even if $\Delta \ll E_F$.