CPC(HEP & NP), 2011, **35**(7): 616–621 Chinese Physics C Vol. 35, No. 7, Jul., 2011

A new parametric equation of state and quark stars*

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Abstract: It is still a matter of debate to understand the equation of state of cold matter with supra-nuclear density in compact stars because of unknown non-perturbative strong interaction between quarks. Nevertheless, it is speculated from an astrophysical view point that quark clusters could form in cold quark matter due to strong coupling at realistic baryon densities. Although it is hard to calculate this conjectured matter from first principles, one can expect that the inter-cluster interaction will share some general features with the nucleon-nucleon interaction successfully depicted by various models. We adopt a two-Gaussian component soft-core potential with these general features and show that quark clusters can form stable simple cubic crystal structure if we assume that the wave function of quark clusters have a Gaussian form. With this parametrization, the Tolman-Oppenheimer-Volkoff equation is solved with reasonably constrained parameter space to give mass-radius relations of crystalline solid quark stars. With baryon number densities truncated at $2n_0$ at surface and the range of the interaction fixed at 2 fm we can reproduce similar mass-radius relations to that obtained with bag model equations of state. The maximum mass ranges from $\sim 0.5M_{\odot}$ to $\gtrsim 3M_{\odot}$. The recently measured high pulsar mass ($\gtrsim 2M_{\odot}$) is then used to constrain the parameters of this simple interaction potential.

Key words: quark star, solid quark matter, mass radius relation, massive pulsar

PACS: 21.65.Qr, 97.60.Gb, 97.60.Jd **DOI:** 10.1088/1674-1137/35/7/004

1 Introduction

An understanding of cold quark matter is both one of the most challenging problems in particle physics and a prerequisite to understand the true nature of pulsars and pulsar-like objects. However, due to both the non-perturbative nature of the strong interaction at low energy and the complexity presented by the quantum many-body problem, it is almost impossible to understand such states theoretically from first principles.

Over the decades, various approaches to bypass these difficulties have been developed, both perturbatively, such as color super-conductivity [1], and non-perturbatively, such as lattice QCD and QCD-based effective models. On the other hand, it has been conjectured [2–4] that quark matter could be in a solid state at extremely low temperature present in the pulsar interior. This possibility could combine naturally with several previous works, suggesting the possibility that deconfined quark matter might contain quark clusters of $3N$ valence quarks [5–7] into a reasonable conjecture that quark clusters could form crystal lattices.

Because of the difficulty in obtaining details of the interaction between quark clusters, it is interesting to apply simple phenomenological models. If we can use astronomical observations to constrain parameters in such models, we may be be able to gain some insights into the properties of low-energy QCD or rule out such forms of cold quark matter within pulsars.

In several previous works [8, 9], different models have been tried to investigate the possible equation of state of solid quark matter and have provided the possibility of explaining stiffness in the equation of state required by observed massive pulsars [10]. In Ref. [9], the Lennard-Jones potential, which was introduced to model the interaction between inert gas molecules [11], is used as the potential between quark clusters. The Lennard-Jones potential shares some

Received 25 September 2010

* Supported by China Scholarship Council, NSFC (10973002, 10935001), National Basic Research Program of China (2009CB824800) and the John Templeton Foundation

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common properties with nuclear forces, such as short-range repulsion and longer range attraction (c.f. [12]). In this article, we adopt a more realistic parametrization that is very similar to various models depicting hyperon-hyperon potential. It has been shown that the interaction between two H-dibaryons – a hypothetical particle with 6 valence quarks [13] may also share this general feature [14].

This article is arranged as follows. The model of inter-cluster potential is presented in Section 2. Parameter space used for calculation is discussed in Section 3. Section 4 shows the result of calculation. The conclusion and some discussions are presented in Section 5.

2 Inter-cluster potential

As in Ref. [9], we consider quark clusters with $3N$ valence quarks with $N=1,6$ and a simple cubic lattice structure is adopted for simplicity. In the context of strange quark matter, these are particles with the same valence quark composition as hyperons and the hypothetical 'quark-alpha' [15]. By extension of the Lennard-Jones potential, we adopt the following non-relativistic interaction potential between two quark clusters located on two lattice sites,

$$v(r)=V_1\mathrm{e}^{-\left(\frac{r}{r_1}\right)^2}-V_2\mathrm{e}^{-\left(\frac{r}{r_2}\right)^2}, \tag{1}$$

where possible spin-dependent terms are omitted for simplicity. As mentioned above, with the condition $V_1>V_2$, $r_1<r_2$, this potential shares the general feature of various successful phenomenological potentials of nuclear interactions: soft-core repulsion at short range and attraction at longer range.

It turned out in later analysis that the maximum masses and the mass-radius curves are not sensitive to the value of r_2. Therefore we fix r_2 to 2 fm, which is a typical range of nuclear force.

For simplicity, we adopt a Gaussian wave packet with width w

$$\psi_{\boldsymbol{r}_0,w}(\boldsymbol{r})=\frac{1}{\pi^{3/4}w^{3/2}}\mathrm{e}^{-\frac{|\boldsymbol{r}-\boldsymbol{r}_0|^2}{2w^2}}, \tag{2}$$

as the wave function of a quark cluster. In Ref. [9], it is assumed that the potential well created by surrounding clusters with Lennard-Jones interaction is deep enough to trap quark clusters at lattice sites. To ensure that the soft-core potential can also trap quark clusters, we adopt a variational method, i.e. determining the value of w by minimizing the total energy of a cluster. The latter is calculated by summing the kinetic energy of the wave packet and the potential energy of nearby clusters. The result shows that with the range of parameters considered in this work, the widths of wave packets are much smaller than the inter-cluster distance. Therefore it makes sense to view the system as consisting of quark clusters trapped at lattice sites. With this small width, the overlap between adjacent wave packets is also negligible. Thus it is reasonable to omit the difference between fermionic and bosonic quark clusters.

To calculate the total contribution to the potential energy, a sum is taken over a cube of 21^3 lattices centered on the quark cluster under consideration. The size of the cube is enough since the cluster number density in this work will not exceed $\sim 20n_0$. Hence, the the total contribution to the potential energy per cluster is

$$\frac{1}{2}V(n)$$
$$\equiv\frac{1}{2}\left(\sum_{k_1=-10}^{10}\sum_{k_2=-10}^{10}\sum_{k_3=-10}^{10}\right)'\tilde{v}\left(w;\frac{\sqrt{k_1^2+k_2^2+k_3^2}}{n^{1/3}}\right), \tag{3}$$

where the prime means that the sum omits $k_1=k_2=k_3=0$, and

$$\tilde{v}(w;r)=\int\mathrm{d}^3\boldsymbol{r}'\psi^*_{0,w}(\boldsymbol{r}')v(r')\psi_{\mathrm{r},w}(\boldsymbol{r}'), \tag{4}$$

is the expectation value of potential operator $v(r)$.

In addition to potential energy, kinetic energy and mass, it is also necessary to include lattice vibration energy in the total energy density. This is considered in the Debye's approximation [16], where the phonons are viewed as a free boson gas with linear dispersion relations $\omega_\parallel=c_\parallel k,\omega_\perp=c_\perp k$ (where $c_\parallel$ and $c_\perp$ are sound velocity for longitudinal and transverse waves) and a cut-off frequency ω_D. It is convenient to define

$$\frac{1}{c_\mathrm{s}^3}=\frac{1}{3}\left(\frac{1}{c_\parallel^3}+\frac{2}{c_\perp^3}\right), \tag{5}$$

and we have

$$\begin{aligned}g(\omega)&=\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left(\delta(\omega-c_\parallel k)+2\delta(\omega-c_\perp k)\right)\\&=\frac{3}{2\pi^2c_\mathrm{s}^3}\omega^2.\end{aligned} \tag{6}$$

The Debye cut-off frequency $\omega_\mathrm{D}=(6\pi^2n)^{1/3}c_\mathrm{s}$ is determined by matching the total number of degrees of freedom $3N$, where N is the number of lattices,

$$V\int_0^{\omega_\mathrm{D}}g(\omega)\mathrm{d}\omega=3N.$$

Each vibration mode has a zero point energy $\frac{1}{2}\hbar\omega$

and the total phonon zero point energy is

$$U_0 = \int_0^{\omega_D} 3\frac{1}{2}\hbar\omega g(\omega)d\omega = \frac{9}{8}(6\pi^2)^{1/3}\hbar c_s n^{4/3}. \quad (7)$$

Following [9], we take sound velocity to be one third of the speed of light $c_s = c/3$ and note that this value has very limited influence on the final result. Therefore the total energy density at zero temperature can be written as

$$\epsilon = \frac{n}{2}V(n,w) + nm + \frac{3}{4mw^2} + \frac{9}{8}(6\pi^2)^{1/3}\hbar c_s n^{4/3}, \quad (8)$$

At zero temperature, the pressure can be derived as

$$P = \frac{n^2}{2}\frac{\partial V}{\partial n} + \frac{n^2}{2}\frac{\partial V}{\partial w}\frac{dw}{dn} - \frac{3}{2mw^3}\frac{dw}{dn} + \frac{3}{8}(6\pi^2)^{1/3}\hbar c_s n^{4/3}. \quad (9)$$

With the energy density and pressure, one can establish the equation of state and solve the Tolman-Oppenheimer-Volkoff (TOV) equations to get the mass-radius relations. In practice, it is more convenient to skip the numerical determination of equation of state $P(\rho)$ and instead write the TOV equation in terms of $n(r)$ and $M(r)$,

$$\frac{dn}{dr} = -\frac{G}{r^2 - 2GMr}(P(n) + \epsilon(n))(M + 4\pi r^3 P(n))\left(\frac{dP}{dn}\right)^{-1}, \quad (10)$$

$$\frac{dM}{dr} = 4\pi r^2 \epsilon(n), \quad (11)$$

$$M(0) = 0, \quad n(0) = n_c, \quad (12)$$

where n_c is the central baryon number density. Because quark matter is usually expected to be self-bound at zero pressure (for instance under the bag model equation of state), it is reasonable to simply adopt a truncation baryon number density n_{surf} at the surface. In this work, we adopt $n_{surf} = 2n_0$, where n_0 is the baryon number density of normal nuclear matter.

3 Parameter space and results

As stated above, r_2 is fixed to 2 fm. Therefore we have 4 free parameters: quark cluster mass m; the height of the two Gaussian components in the

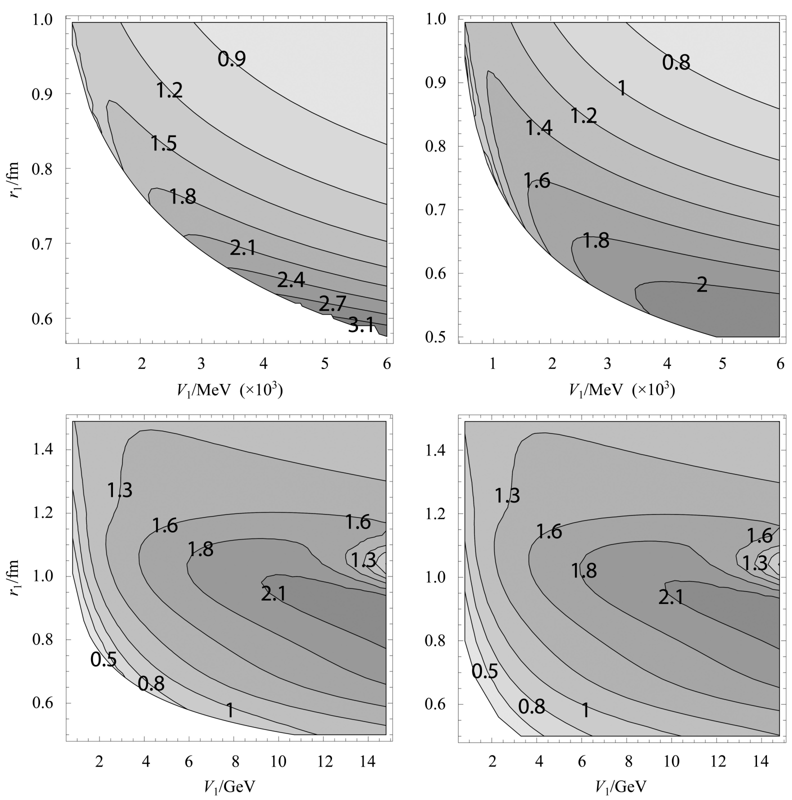

Fig. 1. The maximum mass in units of a solar mass of a solid quark star. upper left: $m = 1$ GeV, $V_2 = 100$ MeV; upper right: $m = 1$ GeV, $V_2 = 50$MeV; lower left: $m = 6$ GeV, $V_2 = 100$ MeV; lower right: $m = 6$ GeV, $V_2 = 50$ MeV. The boundary of contour plots is moved a little higher right than $V_1 r_1^3 = V_2 r_2^3$ to avoid parameters from leading to large error in numerical calculations. It is evident that the maximum mass greater than $2M_\odot$ is allowed in large parameter space.

potential V_1, V_2; and the range of the repulsive core r_1. It is appropriate to expect that the depth of the attractive part of the potential might be of the same order of magnitude as the typical potential between two nucleons in nuclear matter. Therefore we fix V_2 at 50 MeV and 100 MeV, respectively. On the other hand, we fix the value of m to 1 GeV and 6 GeV for 3-quark clusters and the aforementioned hypothetical 'quark-alpha'. Here the mass of the 3-quark cluster is taken from the mass of the Λ hyperon (1115 MeV). Similarly, the mass of the 'quark alpha' (an 18-quark cluster) is assumed to be approximately $6m_\Lambda$. It is worth noting that we made the above choice because in the present work the quark clusters are assumed to be composed of constituent quarks with small mass corrections from potential energy. See Sec. 4 for further discussions. Thus we are left with two free parameters: V_1 and r_1. We adopt a condition,

$$V_1 r_1^3 > V_2 r_2^3, \tag{13}$$

that ensures that the potential energy is always positive (i.e., repulsive) when the density is very high. With the above settings of parameter space, we drew 4 contour plots of maximum mass calculated for 4 different sets of (m, V_2) values, which are shown in Fig. 1.

Typical mass-radius relation curves of these settings with maximum mass exceeding $1.9M_\odot$ are also shown together with corresponding equations of state (Fig. 2, Fig. 3). Stellar mass and radius as functions of central baryon number density n_c are shown in Figs. 4, 5. From Fig. 1, we can see that for our simple soft-core parametrization, maximum mass can range roughly from below $1M_\odot$ to about $\sim 3M_\odot$ for solid quark stars with a 3-quark cluster forming a crystal lattice and from below $0.5M_\odot$ to about $2.1M_\odot$ for solid quark stars made up of 'quark-alpha' particles. On the other hand, typical mass-radius relations shown in Fig. 3 are very similar to those calculated within the bag model equations of state (shown as gray curves in Fig. 3, adopted from model SS1, SS2 in Ref. [17]) and M-R curves calculated in Ref. [9]. This shows that at least for some region of parameter space our simple parametrization can also produce heavy maximum mass supported by the observed value of $2M_\odot$. Inversely, with current observation we can already restrict parameters in this very simple model with only 4 parameters. For instance, to get a maximum mass larger than $1.9M_\odot$ for $m = 1$ GeV and $V_1 < 6$ GeV, we have to restrict r_1 to below 0.75 fm when $V_2 = 100$ MeV and restrict $r_1 \lesssim 0.6$ fm and for $m = 6$ GeV it requires $r_1 \lesssim 1$ fm and $V_1 \gtrsim 8$ GeV.

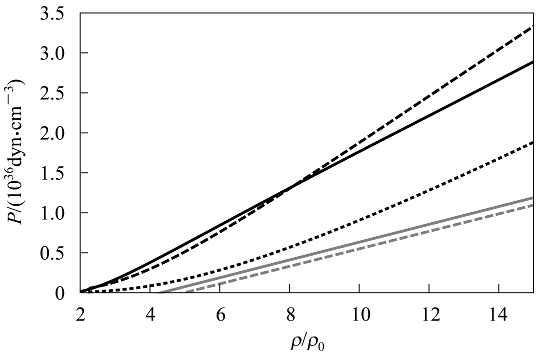

Fig. 2. The equation of state with typical parameters giving large maximum masses. solid: $m = 1$ GeV, $V_1 = 3.2$ GeV, $V_2 = 100$ MeV, $r_1 = 0.68$ fm; dashed: $m = 1$ GeV, $V_1 = 5$ GeV, $V_2 = 50$ MeV, $r_1 = 0.54$ fm; dotted: $m = 6$ GeV, $V_1 = 9$ GeV, $V_2 = 100$ MeV, $r_1 = 0.95$ fm; The gray solid and dashed: bag model equations of state SS1 and SS2 adopted from Ref. [17].

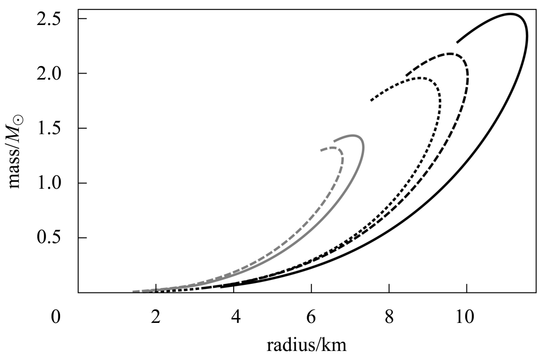

Fig. 3. The mass-radius relation curves for typical parameters giving large maximum masses. The parameter sets for curves are the same as in Fig. 2. Gray curves are calculated with the bag model equation of state.

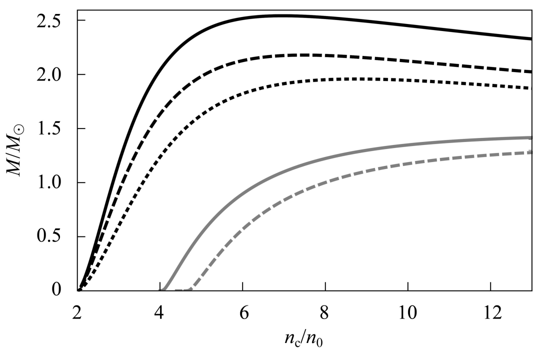

Fig. 4. The stellar mass as a function of central baryon number density n_c for typical parameters. $n_0 = 0.16$ fm^{-3} is the baryon number density at nuclear saturation. The parameter sets for curves are the same as in Fig. 2. Gray curves are calculated with the bag model equation of state.

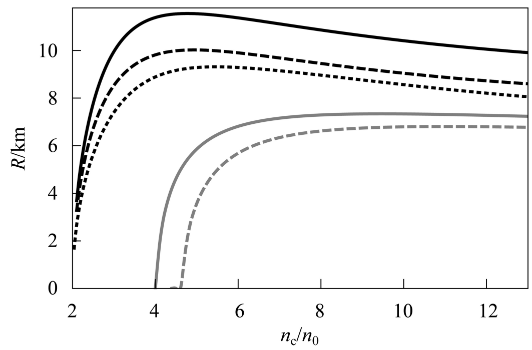

Fig. 5. The stellar radius as a function of central baryon number density n_c for typical parameters. The parameter sets for curves are the same as in Fig. 2. Gray curves are calculated with the bag model equation of state.

4 Discussion

4.1 General discussion

In cold quark matter at baryon number densities realistic for compact stars, the interaction between quarks could be strong enough that instead of condensing in momentum space to form a color-superconductive phase, it is possible that the dressed quarks undergo condensation in position space to form quark clusters. As is stated in Ref. [9], if the potential well formed by neighboring clusters is deep enough to trap each cluster, cold quark matter could form a crystal solid at low temperature.

In this work, we discussed the simple-cubic lattice structure formed by 3- and 18-quark clusters using a simple two-Gaussian-component parametrization of soft-core potential to simulate the interaction between quark clusters. This parametrization shares the basic properties of nucleon-nucleon interaction mediated by meson exchange – short range repulsion, medium and long range attraction and a finite range. These properties are also shared by the Lennard-Jones potential adopted in Ref. [9]. However, unlike the Lennard-Jones potential with r^{-12} pole at a short distance, a soft-core potential adopted in this work can be treated by non-relativistic quantum mechanics using Gaussian wave packets. By minimizing the total energy, we find that at realistic densities this soft-core potential can lead to a stable lattice structure. It is entirely possible that other unit cell structures (e.g. body-centered cubic) are more stable, but we expect that the difference is quantitative instead of qualitative.

Although maximum mass cannot be easily tuned to $\sim 6M_\odot$ as in Ref. [9] due to the soft-core nature of the interaction, our parametrization can still provide a stable lattice crystal structure with maximum mass exceeding $2M_\odot$, which is in accordance with the observed maximum mass [10]. Inversely, the observed maximum mass can be used to put constraints on parameters of this simple model that will possibly give some insights into the form of interaction between quark clusters if such a phase exists.

4.2 Difference between nuclear matter, ordinary quark matter and quark-clustering matter

Another issue needing to be clarified is the difference between this quark-clustering matter and ordinary nuclear matter. Concerning the stellar structure, the biggest difference comes from the existence of a crust. Whether or not a crust is present can severely affect the radius of the quark star with very low mass. As is commonly accepted [18], a neutron star must have a crust where nuclear matter will be substituted by matter with ordinary nuclei while for a quark star it is not necessary to have such restriction. It can have a bare surface, as is assumed in our work with a truncation density, especially if Witten's conjecture [19] is correct, i.e. strange quark matter is at absolute ground state.

On the other hand, we consider the possibility that the quark-clustering phase arises from strong quark-quark interaction. This should be different from confinement or mixed phase of confining and deconfined matter.

At a first look, it would be tempting to extrapolate this model to nuclear saturation density and use the binding energy and bulk modulus to constrain the model parameters in order to improve the accuracy of the prediction. Unfortunately, it would be inappropriate to do this, for three reasons:

1) The quark-clustering phase considered in this article could be in deconfined phase while nuclear matter is a confined phase.

2) The quark-clustering phase can have large strangeness per baryon while saturated nuclear matter does not have such a property.

3) Although the interaction between quark-clusters and that between nucleons are expected to be similar, the major contribution to pressure is different. As is mentioned in Section 3, the overlap between Gaussian wave packets of adjacent quark-clusters is small, so the main pressure contribution comes from the repulsive core of the inter-cluster potential, which is very different from saturated nuclear matter.

It is also worth mentioning that despite great similarity between the mass-radius relation obtained with the bag model equation of state and those calculated here, the underlying picture is drastically different. In the bag model, a quark star is degenerate Fermi gas of free quarks sustained by vacuum energy and associated negative pressure. In this work and Ref. [9], pressure is mainly provided by the repulsive core of the inter-cluster potential instead of mere degenerate pressure.

The quark-clustering matter is also different from ordinary quark matter in chiral symmetry. In the quark-clustering phase, we considered constituent quarks forming clusters of 3 and 18 quarks. In the quark clustering phase, it was conjectured [3, 4] that clusters are formed by dressed quarks. In the present work, the masses of these clusters are chosen simply as the sum of constituent quark masses. The mass of clusters should obviously receive corrections from interaction between constituent quarks, but it is assumed in this work that such corrections are small and will not have a qualitative influence on the result. On the other hand, it was proposed (e.g. in the quark-mass-density-dependent model [20]) that constituent quark masses and current quark masses can be connected by introducing the dependence of quark mass on chemical potential. The influence of this dependence is partly considered in Ref. [21]. It is reasonable to take this dependence into account as part of our future work.

We wish to thank the Pulsar Group at Peking University for useful discussions.

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